SEPARABLE CUBIC STOCHASTIC OPERATORS

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A cubic stochastic operator (CSO) has meaning of a population evolution operator, which arises as follows: Consider a population consisting of m species.

Let $x^{(0)} = (x_1^{(0)}, ..., x_m^{(0)})$ be the probability distribution of species in the initial generations, and $P_{ijk,l}$ the probability that individuals in the ith, jth and kth species interbreed to produce an individual l. Then the probability distribution $x' = (x'_1, ..., x'_m)$ of the species in the first generation can be found by the total probability i.e.

$$W : x'_l = \sum_{i,j,k=1}^{m} P_{ijk,l} x_i^0 x_j^0 x_k^0, \quad l \in E = \{1, ..., m\},$$

where the a matrix $\mathbf{P} \equiv \mathbf{P}(W) = \{P_{ijk,l}\}_{ijk,l=1}^{m}$ satisfying the following properties

$$P_{ijk,l} = P_{kij,l} = P_{ikj,l} = P_{kji,l} = P_{jik,l} = P_{jki,l} \geq 0, \ \sum_{l=1}^{m} P_{ijk,l} = 1 \quad for\ each \quad i, j, k \in E. \tag{1}$$

We define a map W of the simplex

$$S^{m-1} = \left\{ x = (x_1, ..., x_m) \in R^m : x_i \geq 0, \ \sum_{i=1}^{m} x_i = 1 \right\},$$

into itself, by the following rule

$$W : x'_l = \sum_{i,j,k=1}^{m} P_{ijk,l} x_i x_j x_k, \quad l \in E. \tag{2}$$

Definition 1. The operator W (2) is called cubic stochastic operator (CSO).

In this paper we consider CSO (2), (1) with additional properties

$$P_{ijk,l} = a_{il} b_{jl} c_{kl}, \quad for\ all \quad i, j, k, l \in E, \tag{3}$$

where $a_{il}, b_{jl}, c_{kl} \in R$ entries of quadratic matrices $A = (a_{il})$, $B = (b_{jl})$ and $C = (c_{kl})$ such that the properties (1) are satisfied for the coefficients (3).

Then the CSO W corresponding to the matrices A, B and C has the form

$$x'_l = (W(x))_l = (A(x))_l (B(x))_l (C(x))_l, \ \ for\ all \ \ l \in E, \tag{4}$$

where

$$(A(x))_l = \sum_{i=1}^{m} a_{il} x_i, (B(x))_l = \sum_{j=1}^{m} b_{jl} x_j, (C(x))_l = \sum_{k=1}^{m} c_{kl} x_k. \tag{5}$$

Definition 2. The CSO (4) is called separable cubic stochastic operator (SCSO) and we denote it by $W = (A, B, C)$.

We denote by $\mathbf{m}$ quadratic matrix $m \times m$ with elements $m_{ij} = m$, $i, j \in E$

If $A = I_m$ be an identity $m \times m$ matrix, i.e. $a_{il} = 0$ for $i \neq l$ and $a_{ii} = 1$ for all $i, l \in E$, in properties (5). Then the following simple Proposition is useful.

Proposition 3. *Let $A = I_m$, then for matrices $B = (b_{jl})_{j,l=1}^m$ and $C = (c_{kl})_{k,l=1}^m$ of SCSO $W = (I_m, B, C)$ the following property is true: $b_{jl}c_{kl} \geq 0$, $BC^T = \mathbf{m}$ where and C^T is the transpose of C.*

Proposition 4. *If $A = I_3$, $B = (b_{jl})_{j,l=1}^3$ is a skew symmetric matrix. The following equation solvable*

$$B\left(c^{(k)}\right)^T = (3, 3, 3), \quad k = 1, 2, 3 \tag{6}$$

if and only if $b_{23} = b_{13} - b_{12}$. Moreover, for the solution $C = (c_{kl})_{k,l=1}^3$ is the following equality

$$\left(c^{(k)}\right)^T = \left(c_{1k}, \frac{3 + b_{13}c_{1k}}{b_{12} - b_{13}}, \frac{3 + b_{12}c_{1k}}{b_{13} - b_{12}}\right), \quad k = 1, 2, 3 \tag{7}$$

is true, where $(c^{(k)})$ is a row of matrix $C = (c_{kl})_{k,l=1}^3$.

Theorem 5. *If $A = I_3$, $B = (b_{jl})_{j,l=1}^3$ is a skew symmetric matrix and equality (6) is hold, then the SCSO is the quadratic stochastic operator.*

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