A connection between L-index of vector-valued entire function and L-index of each its component

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The present talk is devoted to the properties of entire vector-valued functions of bounded L-index in join variables. We need some notations and definitions. Let $\mathbf{L}:\mathbb{C}^n\to\mathbb{R}^n_+$ be any fixed continuous function. We consider a class of vector-valued entire functions $F=(f_1,\ldots,f_p)\colon\mathbb{C}^n\to\mathbb{C}^p$. For this class of functions there was introduced a concept of boundedness of $\mathbf{L}$-index in joint variables.

Let $\|\cdot\|_0$ be a norm in $\mathbb{C}^p$. Let $\mathbf{L}(z)=(l_1(z),\ldots,l_n(z))$, where $l_j(z)\colon\mathbb{C}^n\to\mathbb{R}_+$ is a positive continuous function. An entire vector-valued function $F\colon\mathbb{C}^n\to\mathbb{C}^p$ is said to be of bounded $\mathbf{L}$-index in joint variables, if there exists $n_0\in\mathbb{Z}_+$ such that $(\forall z=(z_1,\ldots,z_n)\in\mathbb{C}^n)(\forall J\in\mathbb{Z}^n_+)$:

$$\frac{\|F^{(J)}(z)\|_0}{J!\mathbf{L}^J(z)}\le\max\left\{\frac{\|F^{(K)}(z)\|_0}{K!\mathbf{L}^K(z)}\colon K\in\mathbb{Z}^n_+,\|K\|\le n_0\right\},$$

where $F^{(J)}(z)=(f_1^{(J)}(z),\ldots,f_p^{(J)}(z))$, $f_k^{(J)}(z)=\dfrac{\partial^{\|J\|}}{\partial z_1^{j_1}\ldots\partial z_n^{j_n}}f_k(z)$, $\|J\|=j_1+\ldots+j_n$, $J!=j_1!\cdot\ldots\cdot j_n!$ for $J=(j_1,\ldots,j_n)$, $k\in\{1,\ldots,p\}$. The least such integer n_0 is called the $\mathbf{L}$-index in joint variables and is denoted by $N(F,\mathbf{L})$.

Denote by $\mathbb{D}^n[z_0,R/\mathbf{L}(z_0)]=\{z=(z_1,\ldots,z_n)\in\mathbb{C}^n\colon|z_j-z_{j0}|\le r_j/l_j(z_0)\text{ for every }j\in\{1,\ldots,n\}\}$ the closed polydisc in $\mathbb{C}^n$. Let Q^n be a class of continuous functions $\mathbf{L}\colon\mathbb{C}^n\to\mathbb{R}^n_+$ such that $0<\lambda_{1,j}(R)\le\lambda_{2,j}(R)<\infty$ for any $j\in\{1,2,\ldots,n\}$ and $\forall R=(r_1,\ldots,r_n)\in\mathbb{R}^n_+$, where $\lambda_{1,j}(R)=\inf\limits_{z_0\in\mathbb{C}^n}\inf\{l_j(z)/l_j(z_0)\colon z\in\mathbb{D}^n[z_0,R/\mathbf{L}(z_0)]\}$, $\lambda_{2,j}(R)$ is defined analogously with replacement inf by sup.

For $F:\mathbb{C}^n\to\mathbb{C}^p$ let us introduce the sup-norm $|F(z)|_p=\max_{1\le j\le p}\{|F_j(z)|\}$. The notation $A\le B$ for $A=(a_1,\ldots,a_n)$, $B=(b_1,\ldots,b_n)\in\mathbb{R}^n$ means that $a_j\le b_j$ for every $j\in\{1,\ldots,n\}$. The following proposition was firstly deduced for analytic curves in [1]. Similar proposition was also obtained for analytic vector-valued functions $F\colon\mathbb{B}^2\to\mathbb{C}^2$ in the unit ball $\mathbb{B}^2=\{z\in\mathbb{C}^2\colon|z_1|^2+|z_2|^2<1\}$[2]. Here we present it for vector-valued entire functions $F\colon\mathbb{C}^n\to\mathbb{C}^p$.

Proposition 1. *Let $\mathbf{L}=(l_1(z),\ldots,l_n(z))$ be a positive continuous function in $\mathbb{C}^n$. If each component f_s of an entire vector-valued function $F=(f_1,\ldots,f_p)\colon\mathbb{C}^n\to\mathbb{C}^p$ is of bounded $\mathbf{L}$-index $N(\mathbf{L},f_s)$ in joint variables then F is of bounded $\mathbf{L}$-index in joint variables in every norm, in particular, in the sup-norm and $N(\mathbf{L};F)\le\max\{N(\mathbf{L},f_s)\colon 1\le s\le p\}$, and also F is of bounded $\mathbf{L}_*$-index in the Euclidean norm with $\mathbf{L}_*(z,w)\ge\sqrt{p}\mathbf{L}(z,w)$ and $N_E(\mathbf{L}_*,F)\le\max\{N(\mathbf{L},f_s):1\le s\le p\}$. (Here $N(\mathbf{L},F)$ and $N_E(\mathbf{L}_*,F)$ are the $\mathbf{L}$-index and the $\mathbf{L}_*$-index in joint variables with the sup-norm and the Euclidean norm, respectively.)*

Theorem 2 ([3]). *Let $\mathbf{L} \in Q^n$. An entire vector-valued function $F\colon \mathbb{C}^n \to \mathbb{C}^p$ has bounded $\mathbf{L}$-index in joint variables if and only if for every $R \in \mathbb{R}^n_+$ there exist $n_0 \in \mathbb{Z}_+$, $p_0 > 0$ such that for all $z_0 \in \mathbb{C}^n$ there exists $K_0 \in \mathbb{Z}^n_+$, $\|K_0\| \le n_0$, satisfying inequality*

$$\max\Big\{\frac{|F^{(K)}(z)|_p}{K!\mathbf{L}^K(z)} : \|K\| \le n_0, z \in \mathbb{D}^n[z_0, R/\mathbf{L}(z_0)]\Big\} \le p_0 \frac{|F^{(K_0)}(z_0)|_p}{K_0!\mathbf{L}^{K_0}(z_0)}.$$

This theorem is basic in the theory of functions of bounded index. Theorem 2 implies also the following corollary.

Corollary 3. *Let $\mathbf{L} \in Q^n$. An entire vector-function $F\colon \mathbb{C}^n \to \mathbb{C}^p$ has bounded $\mathbf{L}$-index in joint variables in the* sup-norm *if and only if it has bounded $\mathbf{L}$-index in joint variables in the norm $\|\cdot\|_0$.*

Theorem 4. *Let $\mathbf{L} \in Q^n$. In order that an entire vector-valued function $F\colon \mathbb{C}^n \to \mathbb{C}^p$ be of bounded $\mathbf{L}$-index in joint variables it is necessary that for all $R \in \mathbb{R}^n_+$ there exist $n_0 \in \mathbb{Z}_+$, $p_1 \ge 1$ such that for all $z_0 \in \mathbb{C}^n$ there exists $K_0 \in \mathbb{Z}^n_+$, $\|K_0\| \le n_0$, satisfying inequality*

$$\max\{|F^{(K_0)}(z)|_p : z \in \mathbb{D}^n[z_0, R/\mathbf{L}(z_0)]\} \le p_1 |F^{(K_0)}(z_0)|_p \tag{1}$$

and it is sufficiently that for all $R \in \mathbb{R}^n_+$ there exist $n_0 \in \mathbb{Z}_+$, $p_1 \ge 1$ $\forall z_0 \in \mathbb{C}^n$ $\exists K_1^0 = (k_1^0, 0, \ldots, 0)$, $\exists K_2^0 = (0, k_2^0, 0, \ldots, 0), \ldots, \exists K_n^0 = (0, \ldots, 0, k_n^0)$: $k_j^0 \le n_0$, and

$$(\forall j \in \{1, \ldots, n\}):\ \max\{|F^{(K_j^0)}(z)|_p : z \in \mathbb{D}^n[z_0, R/\mathbf{L}(z_0)]\} \le p_1 |F^{(K_j^0)}(z_0)|_p \tag{2}$$

References

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