Invariant structures on homogeneous Φ-spaces and Lie groups

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Homogeneous Φ-spaces were first introduced by V.I. Vedernikov in 1964. Fundamental results for regular Φ-spaces and, in particular, homogeneous k-symmetric spaces were obtained by N.A. Stepanov, A. Ledger, A. Gray, J.A. Wolf, A.S. Fedenko, O. Kowalski and others. It turned out that homogeneous k-symmetric spaces G/H admit a commutative algebra $\mathcal{A}(\theta)$ of *canonical* structures [1]. The remarkable feature of these structures is that all of them are invariant with respect to both the Lie group G and the generalized "symmetries" of G/H. The classical example is the canonical almost complex structure J on homogeneous 3-symmetric spaces with its many applications (N.A. Stepanov, A. Gray, V.F. Kirichenko, S. Salamon and others). For $k > 3$ the algebra $\mathcal{A}(\theta)$ contains a large family of classical structures such as almost complex ($J^2 = -id$), almost product ($P^2 = id$), f-structures of K. Yano ($f^3 + f = 0$) and some others [1]. We dwell on several applications of canonical structures as well as on left-invariant structures on nilpotent and solvable Lie groups.

1) *The generalized Hermitian geometry* (V.F. Kirichenko, D. Blair, S. Salamon and others): canonical nearly Kähler, Killing, Hermitian metric f-structures on homogeneous k-symmetric spaces [2], [3]; left-invariant nearly Kähler and Hermitian f-structures on some classes of nilpotent Lie groups (especially, on 2-step nilpotent and some filiform Lie groups [4]); on generalized (in various senses) Heisenberg groups in dimension 5, 6 [5], and 8; on special solvable Lie groups (group of hyperbolic motions of the plane and its generalizations, the oscillator group and some others); heterotic strings.

2) *Homogeneous Riemannian geometry*: the Naveira classification of Riemannian almost product structures; canonical distributions on Riemannian homogeneous k-symmetric spaces; the classes **F** (foliations), **AF** (anti-foliations), **TGF** (totally geodesic foliations); the Reinhart foliations [6].

3) *Elliptic integrable systems*: homogeneous k-symmetric spaces and associated elliptic integrable systems; a new generalization of almost Hermitian geometry; a new contribution to nonlinear sigma models (F. Burstall, I. Khemar [7]).

4) *Metallic structures*: so-called metallic structures (golden, silver and others), which are fairly popular (especially, golden structures) in many recent publications (M. Crasmareanu, C.-E. Hretcanu [8], A. Salimov, F. Etayo and others); canonical structures of golden type on homogeneous k-symmetric spaces [9].

5) *Symplectic geometry*: bi-Poisson geometry and bi-Hamiltonian systems [10], Hamiltonian vector fields and integrable almost-symplectic Hamiltonian systems [11], canonical almost symplectic structures on Riemannian homogeneous k-symmetric spaces.

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