Commuting sets for topological set operators

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Let X be a set and $F, G : 2^X \to 2^X$ be two set operators on X. We say that a set $A \subset X$ is *commuting set* for the pair F, G if $F(G(A)) = G(F(A))$.

For a topological space X commuting sets for the pair of set operators Cl, Int were characterized by Levine [2] as symmetric differences of clopen sets with nowhere dense sets. Similarly, Staley [3] obtained a criterion for commuting sets for the pair Int, ∂ (here ∂ denotes the topological boundary operator).

In this work we consider the following six set operators on a topological space: Cl, Int, ∂, Ext (the exterior of a set), $*$ and $+$: $A^* = A \backslash IntA$, $A^+ = ClA \backslash A$ (these two operators were explicitly defined and studied by Elez and Papaz [1]). It is possible to obtain characterizations of commuting sets for each pair of these six operators. As an application of these characterizations we present new criteria for the following well-known classes of topological spaces:

- *nodec*: a space in which every nowhere dense set is closed;
- *extremally disconnected*: a space in which the closure of every open set is also open;
- *strongly irresolvable*: a space in which each open subspace is *irresolvable* (i.e. it cannot be expressed as a disjoint union of two dense sets);
- *perfectly disconnected*: a T_0-space in which any pair of disjoint subsets have no common limit points.

Theorem 1. *Let B be a clopen set and C be a nowhere dense set. Then the symmetric difference $B \triangle C$ is a commuting set for the pair $Cl, *$ if and only if $B \cap C$ is closed.*

Corollary 2. *A space is nodec if and only if any commuting set for the pair Cl, Int is also a commuting set for the pair $Cl, *$.*

Proposition 3. *Let X be a space. Then:*

(1) *X is extremally disconnected if and only if any open set is a commuting set for the pair Cl, Int;*
(2) *X is strongly irresolvable if and only if any nowhere dense set is a commuting set for the pair Cl, Int.*

Corollary 4. *A space is extremally disconnected and strongly irresolvable if and only if any set is a commuting set for the pair Cl, Int.*

Proposition 5. *A space is perfectly disconnected if and only if any set is a commuting set for the pair $Cl, *$.*

References

[1] N. Elez and O. Papaz. The new operators in topological space. *Math. Morav.*, 17(2): 63–68, 2013.
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[3] D.H. Staley. On the commutativity of the boundary and interior operators in a topological space. *Ohio J. Sci.*, 68(2): 84, 1968.