Finitely bi-Lipschitz homeomorphisms between Finsler manifolds

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In this talk we investigate the boundary behavior of finitely bi-Lipschitz homeomorphisms between Finsler manifolds. Our study involves the module technique and classes of mappings whose moduli of the curve/surface families are integrally controlled from above and below. The Lusin (N)-property with respect to the k-dimensional Hausdorff measure for the finitely bi-Lipschitz mappings is also established. The talk is based on a joint work with A. Golberg; see [1].

Let $\mathbb{M}$ be an n-dimensional differentiable manifold, $n \geq 2$. By the differentiability we mean C^{∞}–differentiability. For a point $x \in \mathbb{M}$, $T_x\mathbb{M}$ denotes the tangent space at x, and $T\mathbb{M} := \cup_{x \in \mathbb{M}} T_x\mathbb{M}$ is the tangent bundle. The *Finsler manifold* is a differentiable manifold $\mathbb{M}$ equipped the Finsler metric $\Phi(x,\xi) : T\mathbb{M} \to \mathbb{R}^+$ satisfying the conditions:

(i) *regularity*: $\Phi \in C^{\infty}$ on $T\mathbb{M}_0 := T\mathbb{M} \setminus \{0\}$;

(ii) *positive homogeneity*: Φ is positive homogeneous that is $\Phi(x, a\xi) = a\Phi(x,\xi)$ for all positive $a \in R$ and $\Phi(x,\xi) > 0$ for $\xi \neq 0$;

(iii) *the Legendre condition or strong convexity condition*: $g_{ij}(x,\xi) = \frac{1}{2}\frac{\partial^2 \Phi^2(x,\xi)}{\partial \xi^i \partial \xi^j}$ is positive definite whenever $\xi \neq 0$.

Following [3], an element of *volume* on the Finsler manifold is defined by $d\sigma_{\Phi}(x) := \frac{|B^n|}{|B^n_x|}\, dx^1 ... dx^n$, where $|B^n|$ denotes the Euclidean volume of the unit n-ball whereas $|B^n_x|$ is the Euclidean volume of the set $B^n_x = \left\{ (\xi^1, ..., \xi^n) \in \mathbb{R}^n : \Phi\left(x, \sum\limits_1^n (\xi^i, e_i(x))\right) < 1 \right\}$ with an arbitrary basis $\{e_i(x)\}_{i=1}^n$ in $\mathbb{R}^n$ depending on x.

Suppose that D and D' are two domains on $\mathbb{M}$ and $\mathbb{M}'$, respectively, $f : D \to D'$ is a continuous mapping. Let $L(x,f) = \limsup_{y \to x} \frac{d_{\Phi'}(f(x),f(y))}{d_{\Phi}(x,y)}$, $x \in D$ and $l(x,f) = \liminf_{y \to x} \frac{d_{\Phi'}(f(x),f(y))}{d_{\Phi}(x,y)}$.

Following [2], we say that $f : D \to D'$ is *finitely Lipschitz* if $L(x,f) < \infty$ for all $x \in D$ and *finitely bi-Lipschitz* if

$$0 < l(x,f) \leq L(x,f) < \infty$$

for all $x \in D$.

A Borel function $\rho : \mathbb{M} \to [0,\infty]$ is called *admissible* for the family Γ of k-dimensional surfaces S in $\mathbb{M}$, $k = 1, \dots, n-1$, (abbr. $\rho \in \operatorname{adm}\Gamma$), if

$$\int\limits_S \rho^k \, d\mathcal{A}_{\Phi} \geq 1, \qquad \forall\, S \in \Gamma. \tag{1}$$

Following [2], the function $\rho : \mathbb{M} \to [0,\infty]$ measurable with respect to the measure of a volume σ_{Φ} is called *extensively admissible* for a family Γ of k-dimensional surfaces S in $\mathbb{M}$ (abbr. $\rho \in \operatorname{ext\,adm}\Gamma$), if the admissibility condition (1) holds for almost all (a.a.) $S \in \Gamma$.

The *conformal module* or *module* (called also the conformal modulus) of a family Γ of k-dimensional surfaces in D is defined by

$$M(\Gamma) := \inf_{\rho \in \operatorname{adm}\Gamma} \int\limits_D \rho^n(x)\, d\sigma_{\Phi}(x),$$

where D is a domain in $\mathbb{M}$.

Let $Q:\mathbb{M}\to(0,\infty)$ be a measurable function. A homeomorphism $f:D\to D'$ is called *lower Q-homeomorphism* at a point $x_0\in\overline{D}$, if there exists $\delta_0\in(0,d(x_0))$, $d(x_0):=\sup\limits_{x\in D}d_\Phi(x,x_0)$, such that for any $\varepsilon_0<\delta_0$ and any geodesic rings $A_\varepsilon=A(x_0,\varepsilon,\varepsilon_0)=\{x\in\mathbb{M}:\varepsilon<d_\Phi(x,x_0)<\varepsilon_0\}$, $\varepsilon\in(0,\varepsilon_0)$, the inequality

$$M(f(\Sigma_\varepsilon))\ \geq\ \inf_{\rho\in\operatorname{ext}\operatorname{adm}\Sigma_\varepsilon}\int\limits_{D\cap A_\varepsilon}\frac{\rho^n(x)}{Q(x)}\,d\sigma_\Phi(x)\tag{2}$$

holds. Here Σ_ε stands for the family of all intersections of the geodesic spheres $S(x_0,r)=\{x\in\mathbb{M}:d_\Phi(x,x_0)=r\},r\in(\varepsilon,\varepsilon_0)$, with the domain D. We say that the homeomorphism $f:D\to D'$ is a *lower Q-homeomorphism* in D, if f is lower Q-homeomorphism at every point $x_0\in\overline{D}$.

For sets A,B and C, we denote by $\Delta(A,B;C)$ the set of all curves $\gamma:[a,b]\to\mathbb{M}$, which join A and B in C, i.e. $\gamma(a)\in A$, $\gamma(b)\in B$ and $\gamma(t)\in C$ for all $t\in(a,b)$.

Let $Q:\mathbb{M}\to(0,\infty)$ be a measurable function. We say that a homeomorphism $f:D\to D'$ is *ring Q-homeomorphism at a point $x_0\in\overline{D}$*, if

$$M\left(\Delta(f(K),f(K_0);D')\right)\ \leq\int\limits_{D\cap A_\varepsilon}Q(x)\cdot\eta^n\left(d_\Phi(x,x_0)\right)d\sigma_\Phi(x)\tag{3}$$

holds for any geodesic ring $A_\varepsilon=A(x_0,\varepsilon,\varepsilon_0)$, $0<\varepsilon<\varepsilon_0<\infty$, any two continua (compact connected sets) $K\subset\overline{B(x_0,\,\varepsilon)}\cap D$ and $K_0\subset D\setminus B(x_0,\varepsilon_0)$ and each Borel function $\eta:(\varepsilon,\varepsilon_0)\to[0,\infty]$, such that $\int\limits_\varepsilon^{\varepsilon_0}\eta(r)dr=1$. We say that f is *a ring Q-homeomorphism in D*, if (3) holds for all points $x_0\in\overline{D}$.

Recall that a metric space $\mathbb{M}$ is called *hyperconvex* if $\cap_{\alpha\in\Lambda}\overline{B}(x_\alpha,r_\alpha)\neq\emptyset$ for any collection of points $\{x_\alpha\}_{\alpha\in\Lambda}$ in $\mathbb{M}$ and positive numbers $\{r_\alpha\}_{\alpha\in\Lambda}$ such that $d(x_\alpha,x_\beta)\leq r_\alpha+r_\beta$ for any α and β in Λ.

The main result of talk is following

Theorem 1. ([1]) *Let D and D' be two domains in Finsler n-dimensional manifolds $(\mathbb{M},\Phi)$ and $(\mathbb{M}',\Phi')$, respectively, $n\geq 2$, and let $\mathbb{M}'$ be a hyperconvex space. If $f:D\to D'$ is a finitely bi-Lipschitz homeomorphism then f is both lower Q-homeomorphism with $Q=K_I^{\frac{1}{n-1}}(x,f)$ and ring Q_*-homeomorphism with $Q_*=C\cdot K_I(x,f)$, where $K_I(x,f)\in L^1_{loc}$ stands for the inner dilatation of mapping f, and C is a constant arbitrarily close to 1.*

References

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[3] Z. Shen. On Finsler geometry of submanifolds. *Math. Ann.*, 311 (3): 549–576, 1998.