Some topological obstructions for strong coloring of uniform hypergraphs

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A hypergraph $H = (V, E)$ based on the vertex set V and with the edge set E is called k-uniform if all its edges have cardinality k. A strong l-coloring of the hypergraph H is a map $h\colon V \to [l]$ where $[l] = \{1, 2, \ldots, l\}$ such that for each edge $e = \{v_1, v_2, \ldots, v_k\} \in E$ the vertices $v_1, v_2, \ldots, v_k$ are labeled with different colors. Note that a strong coloring of a uniform hypergraph H is just a proper coloring of its 1-skelethon $H^{(1)}$, which is covered by a collection of k-cliques.

Let $\mathbf{S}$ be a family of nonempty subsets of some base set X. The generalized Kneser hypergraph $Kg^k_m(\mathbf{S})$ where $m \leq k-1$ is defined as follows. The vertices of $Kg^k_m(\mathbf{S})$ are the elements S_i of $\mathbf{S}$ and there is a k-edge $e = \{S_{i_1}, \ldots, S_{i_k}\}$ in $Kg^k_m(\mathbf{S})$ if and only if $S_{i_1} \cap \cdots \cap S_{i_{m+1}} = \emptyset$ for any distinct sets $S_{i_1}, \ldots, S_{i_{l+1}}$ from $\mathbf{S}$ (see also [1, 2]).

In the present talk, we represent k-uniform hypergraphs H as generalized Kneser hypergraphs $Kg^k_{k-1}(\mathbf{S})$. For the given k, l with $l \geq k$ we define the generalized Kneser k-uniform hypergraph $Kg^k_{k-1}(\mathbf{T})$ which is called the testing hypergraph for l-coloring of k-uniform hypergraphs. Both $Kg^k_{k-1}(\mathbf{S})$ and $Kg^k_{k-1}(\mathbf{T})$ have the natural geometric interpretation as cell complexes, denoted by $B_k(Kg^k_{k-1}(\mathbf{S}))$ and $B_{k,l}(\mathbf{T})$, respectively. The cell complexes $B_k(Kg^k_{k-1}(\mathbf{S}))$ and $B_{k,l}(\mathbf{T})$ are enhanced with natural action of the symmetric grup S_k. The action of the group S_k is effective on both cell complexes. For each l-coloring of a k-uniform hypergraph H there is a natural homomorfizm $g\colon Kg^k_{k-1}(\mathbf{S}) \to Kg^k_{k-1}(\mathbf{T})$ of hypergraphs $Kg^k_{k-1}(\mathbf{S})$ and $Kg^k_{k-1}(\mathbf{T})$. The homomorfizm $g\colon Kg^k_{k-1}(\mathbf{S}) \to Kg^k_{k-1}(\mathbf{T})$ induces an S_k-equivariant cellular map $g'\colon B_k(Kg^k_{k-1}(\mathbf{S})) \to B_{k,l}(\mathbf{T})$. Therefore, the nonexistence of such S_k-equivariant map from $B_k(Kg^k_{k-1}(\mathbf{S}))$ to $B_{k,l}(\mathbf{T})$ is a topological obstruction for existence of strong l-coloring of the k-uniform hypergraph H. We discuss the conditions under which such topological obstructions do not vanish.

References

[1] C.E.M.C. Lange, and G. M. Ziegler, *Note on generalized Kneser Hypergraph coloring* , J. Comb. Theory, ser. A **114**, 2007, pp. 159-166.

[2] G. M. Ziegler, *Generalized Kneser coloring theorems with combinatorial proofs*, Inventiones Math., **147**, 2002, pp.671-–691.