Ricci-flat Kähler metrics on tangent bundles of rank-one symmetric spaces of compact type

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We give an explicit description of all complete G-invariant Ricci-flat Kähler metrics on the tangent bundle $T(G/K) \cong G^{\mathbb{C}}/K^{\mathbb{C}}$ of rank-one Riemannian symmetric spaces G/K of compact type, in terms of associated vector-functions.

Over the latest decades there has been considerable interest in Ricci-flat Kähler metrics whose underlying manifold is diffeomorphic to the tangent bundle $T(G/K)$ of a Riemannian symmetric space G/K of compact type. For instance, a remarkable class of Ricci-flat Kähler manifolds of cohomogeneity one was discovered by M. Stenzel [1]. This has originated a great deal of papers. To cite but a few: M. Cvetič, G. W. Gibbons, H. Lü and C. N. Pope [2] studied certain harmonic forms on these manifolds and found an explicit formula for the Stenzel metrics in terms of hypergeometric functions. Earlier, T. C. Lee [3] gave an explicit formula of the Stenzel metrics for classical spaces G/K but in another vein, using the approach of G. Patrizio and P. Wong [4]. Remark also that in the case of the standard sphere $\mathbb{S}^2$, the Stenzel metrics coincide with the well-known Eguchi-Hanson metrics [5]. On the other hand, and as it is well known, Stenzel metrics continue being a source of results both in physics and differential geometry. We cite here only to G. Oliveira [6] and M. Ionel and T. A. Ivey [7].

We give an *explicit* description of all complete G-invariant Ricci-flat Kähler metrics on the tangent bundle $T(G/K)$ of rank-one Riemannian symmetric spaces G/K of compact type or, equivalently, on the complexification $G^{\mathbb{C}}/K^{\mathbb{C}}$ of G/K. To this end, we use the method of our article [8], giving the result in terms of associated vector-functions (see below). It is also shown that this set of metrics contains a new family of metrics which are not $\partial\bar{\partial}$-exact if $G/K \in \{\mathbb{C}\mathbf{P}^n, n \geqslant 1\}$, and coincides with the set of $\partial\bar{\partial}$-exact Stenzel metrics for any of the latter spaces G/K.

Remark here that until now, in the case of the space $\mathbb{C}\mathbf{P}^n$ ($n \geqslant 1$), all known Ricci-flat Kähler metrics were Calabi metrics, so being hyper-Kählerian and thus automatically Ricci-flat (see O. Biquard and P. Gauduchon [9, 10] and E. Calabi [11]). Since by A. Dancer and M.Y. Wang [12, Theorem 1.1] any complete G-invariant hyper-Kählerian metric on $G/K = \mathbb{C}\mathbf{P}^n$ ($n \geqslant 2$) coincides with the Calabi metric, our new metrics are not hyper-Kählerian.

Note also, that in [12] the Kähler-Einstein metrics on manifolds of G-cohomogeneity one were classified but only under one additional assumption: It is assumed that the isotropy representation of the space G/H (see our notation below) splits into pairwise inequivalent sub-representations. This condition is crucial for the fact that the Einstein equation can be solved. But this assumption fails, for instance, for the symmetric space $\mathbb{C}\mathbf{P}^n$ ($n \geqslant 2$).

Let G/K be a rank-one symmetric space of a compact connected Lie group G. The tangent bundle $T(G/K)$ has a canonical complex structure J_c^K coming from the G-equivariant diffeomorphism $T(G/K) \to G^{\mathbb{C}}/K^{\mathbb{C}}$. The latter space is the above-mentioned complexification of G/K. In our paper [8] we described, for such a G/K, all G-invariant Kähler structures $(\mathbf{g}, J_c^K)$ which are moreover Ricci-flat on the punctured tangent bundle $T^+(G/K)$ of $T(G/K)$. This description is based on the fact that $T^+(G/K)$ is the image of $G/H \times \mathbb{R}^+$ under certain G-equivariant diffeomorphism. Here H denotes the stabilizer of any element of $T(G/K)$ in general position. Such G-invariant Kähler and Ricci-flat Kähler structures are determined completely by a unique vector-function $\mathbf{a}\colon \mathbb{R}^+ \to \mathfrak{g}_H$ satisfying certain conditions, $\mathfrak{g}_H$ being the subalgebra of $\mathrm{Ad}(H)$-fixed points of the Lie algebra of G.

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