Spaces of probability measures and box dimension

Natalia Mazurenko
(Department of Mathematics and Computer Science, Vasyl Stefanyk Precarpathian National University, Shevchenka Str., 57, Ivano-Frankivsk,76025, Ukraine.)
E-mail: `mnatali@ukr.net`

Mykhailo Zarichnyi
(Department of Mechanics and Mathematics, Lviv National University, Universytetska Str., 1, Lviv, 79000, Ukraine)
E-mail: `zarichnyi@yahoo.com`

The topology of hyperspaces of sets of given dimension is investigated in numerous publications (see, e.g., [3, 5, 2, 1]). In particular, in [7] the author described the topology of the hyperspace of sets of given Hausdorff dimension.

There are different definitions of dimension for the probability measures (see, e.g., [4, 10, 6]). The present talk is devoted to the spaces of probability measures of given box dimension.

Let us recall some necessary definitions.

Let (X, d) be a complete metric space. For $F \subset X$ and $r > 0$ denote by $\mathcal{N}_r(F)$ the least number of closed balls of radius r needed to cover the set F. The *lower* and *upper box dimensions* of a set F are defined as follows:

$$\underline{\dim}_{box}(F) = \underline{\lim}_{r\to 0}\frac{\log \mathcal{N}_r(F)}{\log(1/r)};$$
$$\overline{\dim}_{box}(F) = \overline{\lim}_{r\to 0}\frac{\log \mathcal{N}_r(F)}{\log(1/r)},$$

and the *box dimension* of a set F is defined as

$$\dim_{box}(F) = \lim_{r\to 0}\frac{\log \mathcal{N}_r(F)}{\log(1/r)}$$

whenever the last limit exists.

Note also that the lower (upper) box dimension of a set coincides with the lower (upper) box dimensions of its topological closure.

Additionally, let the quantity λ_0 be given by the formula

$$\lambda_0 = \lambda_0(X) = \inf\{\overline{\dim}_{box}(B_r(x)) \mid x \in X, r > 0\},$$

where $B_r(x)$ denotes the closed ball of radius r centered at x, is called the *smallest local upper box dimension* of X.

Now, let $P(X)$ denote the space of probability measures on X and $k > 0$. We endow $P(X)$ with the weak* topology. The *lower* and *upper box dimensions* of a measure $\mu \in P(X)$ are defined, respectively, by the formulae:

$$\underline{\dim}_{box}(\mu) = \lim_{k\to 0}\inf\{\underline{\dim}_{box}(F) \mid F \in \mathcal{B}(X),\ \mu(F) \geq 1-k\};$$
$$\overline{\dim}_{box}(\mu) = \lim_{k\to 0}\inf\{\overline{\dim}_{box}(F) \mid F \in \mathcal{B}(X),\ \mu(F) \geq 1-k\},$$

where $\mathcal{B}(X)$ denotes the set of Borel subsets in X.

Theorem 1. *Let X be an infinite complete separable metric space, then the set*

$$\{\mu\colon \underline{\dim}_{box}(\mu) = 0\}$$

is homeomorphic to the separable Hilbert space l^2.

Theorem 2. *Let X be an infinite complete separable metric space, then the set*

$$\{\mu\colon \inf\{\overline{\dim}_{box}(F)\colon F \in \mathcal{B}(X), \mu(F) > 0\} \geq \lambda_0\}$$

is homeomorphic to the separable Hilbert space l^2.

The proofs of these theorems are based on results by Myjak and Rudnicki [9]. They proved that the mentioned sets are residual in the space of probability measures. We also apply some characterization theorems from infinite-dimensional topology [8].

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