Automorphisms of cellular divisions of 2-sphere induced by functions with isolated critical points

Anna Kravchenko
(Taras Shevchenko National University of Kyiv, Ukraine)
E-mail: annakravchenko1606@gmail.com

Sergiy Maksymenko
(Institute of Mathematics, National Academy of Sciences of Ukraine, Kyiv, Ukraine)
E-mail: maks@imath.kiev.ua

In general, if $f : M \to \mathbb{R}$ is an arbitrary smooth function with isolated critical points, then a certain part of its "combinatorial symmetries" is reflected by a so-called *Kronrod-Reeb* graph Δ_f, see e.g. [6, 2, 5, 4, 14, 13, 12, 1]. Such a graph is obtained by shrinking each connected component of each level set $f^{-1}(c)$, $c \in \mathbb{R}$, of f into a point.

Let $\mathcal{D}(M)$ the group of diffeomorphisms of M and

$$\mathcal{S}(f) = \{h \in \mathcal{D}(M) \mid f(h(x)) = f(x) \text{ for all } x \in M\}$$

be the group of diffeomorphisms h of M which "preserve" f in the sense that h leaves invariant each level set $f^{-1}(c)$, $c \in \mathbb{R}$, of f. Hence it yields a certain permutation of connected components of $f^{-1}(c)$ being points of Δ_f, and thus induces a certain map $\rho(h) : \Delta_f \to \Delta_f$. It can be shown that $\rho(h)$ is a homeomorphism of Δ_f, and the correspondence $\rho : h \mapsto \rho(h)$ is a *homomorphism* of groups

$$\rho : \mathcal{S}(f) \to \mathcal{H}(\Delta_f),$$

where $\mathcal{H}(\Delta_f)$ is the group of homeomorphisms of Δ_f. One can also verify that the image of $\rho(\mathcal{S}(f))$ is a *finite* group.

Let also $\mathcal{D}_{id}(M)$ be the identity path component of $\mathcal{D}(M)$, and

$$\mathcal{S}'(f) = \mathcal{S}(f) \cap \mathcal{D}_{id}(M)$$

be the group of f-preserving diffeomorphisms which are isotopic to the identity via an isotopy consisting of not necessarily f-preserving diffeomorphsms. We will be interested in the group

$$G_f = \rho(\mathcal{S}'(f))$$

of automorphisms of Δ_f induced by elements from $\mathcal{S}'(f)$.

Suppose that the set $\text{Fix}(G_f)$ of common fixed points of all elements of G_f in Δ_f is non-empty. Let also $v \in \text{Fix}(G_f)$ be a vertex of Δ_f fixed under G_f and $Star(v)$ be a *star* of v, i.e. a small G_f-invariant neighborhood of v. Then each $\gamma \in G_f$ induces a homeomorphism of $Star(v)$, and we can also define the group

$$G_v^{loc} = \{\gamma|_{Star(v)} \mid \gamma \in G_f\}$$

of restrictions of elements of G_f to $Star(v)$. We will call G_v^{loc} the *local stabilizer* of v.

Remark 1. We will give now an equivalent description of the group G_v^{loc}. Let K be the critical component of a level-set of f corresponding to the vertex $v \in \Delta_f$. Since $v \in \text{Fix}(G_f)$, we obtain that $h(K) = K$ for all $h \in \mathcal{S}'(f)$. Let $c = f(K)$ be the value of f on K, and $\varepsilon > 0$ be a small number such that the segment $[c-\varepsilon, c+\varepsilon]$ contains no other critical values of f except for c. Let also N_K be the connected component of $f^{-1}[c-\varepsilon, c+\varepsilon]$ containing K. Notice that the quotient map p induces a bijection between connected components ∂N_K and edges of $Star(v)$. Moreover, $h(N_K) = N_K$ for all $h \in \mathcal{S}'(f)$, and hence h induces a permutation σ_h of connected components of ∂N_K. Then G_v^{loc} is the same as the group of permutations of connected components of ∂N_K induced by h.

In [9, 7, 8, 10, 11], the groups G_v^{loc} were calculated for all Morse functions on all orientable surfaces distinct from S^2. In the present paper, we give a complete description of the structure of the group G_v^{loc} to the case when $M = S^2$. For the convenience of the reader we present a general statement about the structure of the group G_v^{loc} for all orientable surfaces.

Theorem 2. *Let $f \in C^\infty(M, \mathbb{R})$ be a Morse function and $v \in \mathrm{Fix}(G_f)$ be some vertex.*

(1) *If $M \neq S^2, T^2$, then $G_v^{loc} \approx \mathbb{Z}_n$, for some $n \geq 1$,* [9].

(2) *If $M = T^2$, then $G_v^{loc} \approx \mathbb{Z}_m \times \mathbb{Z}_{mn}$, for some $m, n \geq 1$,* [7, 8, 10].

(3) *Let $M = S^2$. Then the following statements hold.*

(a) *For each vertex $v \in \mathrm{Fix}(G_f)$, the group G_v^{loc} is isomorphic to a finite subgroup of $SO(3)$, that is, to one of the following groups, see* [3, pp. 21-23]:

$$\mathbb{Z}_n, \quad \mathbb{D}_n, \quad \mathbb{A}_4, \quad \mathbb{S}_4, \quad \mathbb{A}_5, \quad (n \geq 1). \tag{1}$$

(b) *If $\mathrm{Fix}(G_f)$ has at least one edge, then for any vertex $v \in \mathrm{Fix}(G_f)$, the group G_v^{loc} is cyclic.*

(c) *If $\mathrm{Fix}(G_f)$ consists of a unique vertex v and G_v^{loc} is non-trivial and cyclic, then $G_v^{loc} \cong \mathbb{Z}_2$.*

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