Mixed volumes/areas and distribution of zeros of holomorphic functions

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We denote by $\mathbb{N} := \{1, 2, \dots\}$, $\mathbb{R}$, $\mathbb{R}^+ := \{x \in \mathbb{R} \colon x \geq 0\}$, $\mathbb{R}^+_* := \mathbb{R}^+ \setminus 0$, and $\mathbb{C}$ the sets of *natural*, of *real*, of *positive*, of *strictly positive*, and of *complex* numbers, each endowed with its natural order ($\leq$, $\sup / \inf$), algebraic, geometric and topological structure.

Mixed areas/volumes. Let S be a bounded subset in $\mathbb{C}$ with the *support function* [1, Ch. 1], [2]

$$\mathsf{Sp}_S(z) \underset{z\,\in\,\mathbb{C}}{:=} \sup_{w\in S} \operatorname{Re}(z\bar{w}), \quad \mathsf{sp}_S(t) \underset{t\,\in\,\mathbb{R}}{:=} \mathsf{Sp}_S(e^{it}), \text{ and } \quad \Delta_S(t) \underset{t\,\in\,\mathbb{R}}{:=} (\mathsf{sp}_S)'_{\text{left}}(t) + \int_0^t \mathsf{sp}_S(x)\,\mathrm{d}x,$$

where $(\mathsf{sp}_S)'_{\text{left}}$ is the left derivative of sp_S. The *mixed area (of Minkowski)* $\mathsf{F}(S_1, S_2)$ of bounded sets $S_1, S_2 \subset \mathbb{C}$ is integrals [1, Ch. 1, 3], [2], [3, § 4]

$$\mathsf{F}(S_1, S_2) := \frac{1}{2}\int_0^{2\pi} \mathsf{sp}_{S_1}(t)\,\mathrm{d}\Delta_{S_2}(t) = \frac{1}{2}\int_0^{2\pi} \Bigl(\mathsf{sp}_{S_1}\mathsf{sp}_{S_2} - (\mathsf{sp}_{S_1})'_{\text{left}}(\mathsf{sp}_{S_2})'_{\text{left}}\Bigr)(t)\,\mathrm{d}t = \mathsf{F}(S_2, S_1).$$

Convexity with respect to a pair of functions. Let $I \subset \mathbb{R}$ be an open interval, and let $f_1 \colon I \to \mathbb{R}$ and $f_2 \colon I \to \mathbb{R}$ be a pair of functions. A function $g \colon I \to \mathbb{R}$ will be called *convex with respect to the pair* f_1, f_2, or, briefly, (f_1, f_2)*-convex* if there is a number $d > 0$ such that for each $x_1, x_2 \in I$ with $|x_1 - x_2| < d$ and for each $C_1, C_2 \in \mathbb{R}$ such that

$$g(x_1) \leq C_1 f_1(x_1) + C_2 f(x_1), \quad g(x_2) \leq C_1 f_1(x_2) + C_2 f(x_2),$$

we have $g(x) \leq C_1 f_1(x) + C_2 f(x)$ for each $x \in (x_1, x_2)$ [4, Ch. I, § 1]. So, if $g \colon \mathbb{R}^+_* \to \mathbb{R}^+_*$ is (f_1, f_2)-convex for the pair $f_1 \colon x \underset{x\,\in\,\mathbb{R}^+_*}{\longmapsto} x$ and $f_1 \colon x \underset{x\,\in\,\mathbb{R}^+_*}{\longmapsto} 1/x$, then we say that g is $(x, 1/x)$-convex.

Entire functions in $\mathbb{C}$. Even the following special result develops [3, § 4], [5], [6, Ch. 3, 4.2].

Let $f \neq 0$ be an entire function of exponential type with the indicator of growth of f denoted by

$$\mathsf{Ind}_1[f](z) := \limsup_{0<r\to+\infty} \frac{\ln\bigl|f(rz)\bigr|}{r} \underset{z\,\in\,\mathbb{C}}{\in} \mathbb{R}.$$

The function $\mathsf{Ind}_1[f]$ is convex and positive homogeneous on $\mathbb{C}$. Therefore, there is a non-empty convex compact set $I_f \subset \mathbb{C}$ with $\mathsf{Sp}_{I_f} = \mathsf{Ind}_1[f]$ called the *indicator diagram* of this entire function f.

Theorem 1. *Let $f \neq 0$ be an entire function of exponential type with the indicator diagram I_f. Suppose that the function f vanish on a sequence $\mathsf{Z} = (\mathsf{z}_k)_{k\in\mathbb{N}} \subset \mathbb{C}$, i.e., $f(\mathsf{z}_k) = 0$ for each $k \in \mathbb{N}$. If K is a non-empty convex compact subset in $\mathbb{C}$, and $g \colon \mathbb{R}^+_* \to \mathbb{R}^+_*$ is an increasing $(x, 1/x)$-convex function on $\mathbb{R}^+_*$, such that*

$$0 < \liminf_{0<x\to+\infty} \frac{g(x)}{x} \leq \limsup_{0<x\to+\infty} \frac{g(x)}{x} < +\infty, \tag{1}$$

then

$$\limsup_{1<a\to+\infty}\limsup_{r\to+\infty}\frac{\pi}{\int\limits_r^{ar} g(1/x)\,\mathrm{d}x}\sum_{r<|\mathsf{z}_k|\le ar} g\Bigl(\frac{1}{|\mathsf{z}_k|}\Bigr)\mathsf{sp}_K\Bigl(\frac{\mathsf{z}_k}{|\mathsf{z}_k|}\Bigr)\le \mathsf{F}(I_f,K). \tag{2}$$

Besides, for the identity function $g\colon x \underset{x\,\in\,\mathbb{R}^+_*}{\longmapsto} x$ *and for any non-empty convex compact subsets* S *and* K *in* $\mathbb{C}$*, there is an entire function* $f\neq 0$ *of exponential type with zero sequence* $\mathsf{Z}=(\mathsf{z}_k)_{k\in\mathbb{N}}\subset\mathbb{C}$ *and the indicator diagram* $I_f=S$ *such that we have the equality in* (2).

Theorem 1 can be extended to entire functions of finite order $\rho\in\mathbb{R}^+$ of one or several complex variables with significant generalizations of mixed areas/volumes for ρ-convex sets. Thus, we obtain numerous exact results on the completeness of systems of entire functions in classical function spaces on sets in $\mathbb{C}^n$ for $n\in\mathbb{N}$ in terms of the *mutual indicator* of entire function and set [5], [6, Ch. 3, 4.2].

Here we note only the simplest version of the application of Theorem 1 to completeness questions.

Completeness of exponential systems. For a compact subset S of $\mathbb{C}$, we denote by $C(S)\cap\mathrm{Hol}(\mathrm{int}S)$ the normed space of all continuous functions $f\colon S\to\mathbb{C}$ such that the restriction f to the interior $\mathrm{int}S$ of S is holomorphic if this interior $\mathrm{int}S$ is non-empty, equipped with the norm $\|f\|_S:=\sup\limits_{s\in S}\bigl|f(s)\bigr|$.

Theorem 2. *Let* S *be a compact subset of the complex plane such that* $\mathbb{C}\setminus S$ *is connected. Let* $\mathsf{Z}=(\mathsf{z}_k)_{k\in\mathbb{N}}\subset\mathbb{C}$ *be a sequence of pairwise distinct numbers. If there are a non-empty compact convex subset* $K\subset\mathbb{C}$ *and an increasing* $(x,1/x)$*-convex function* $g\colon\mathbb{R}^+_*\to\mathbb{R}^+_*$ *satisfying* (1) *such that*

$$\limsup_{1<a\to+\infty}\limsup_{r\to+\infty}\frac{\pi}{\int\limits_r^{ar} g(1/x)\,\mathrm{d}x}\sum_{r<|\mathsf{z}_k|\le ar} g\Bigl(\frac{1}{|\mathsf{z}_k|}\Bigr)\mathsf{sp}_K\Bigl(\frac{\bar{\mathsf{z}}_k}{|\mathsf{z}_k|}\Bigr)> \mathsf{F}(I,K).$$

then the closure of the linear hull of $\bigl\{e^{\mathsf{z}_k s}\colon k\in\mathbb{N}\bigr\}$ *in* $C(S)\cap\mathrm{Hol}(\mathrm{int}S)$ *coincides with* $C(S)\cap\mathrm{Hol}(\mathrm{int}S)$.

Holomorphic functions in the unit disk/ball. In [7] and [8], we first used ρ-trigonometrically convex and ρ-subspherical functions to study zero sets of holomorphic functions on the unit disk in $\mathbb{C}$ and on the unit ball in $\mathbb{C}^n$, respectively. Some of these results can be obtained in a more general form in terms of mixed areas/volumes and Hausdorff measures of zero sets.

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