Structure of functions on an oriented 2-manifold with the boundary

Bohdana Hladysh
(Taras Shevchenko National University of Kyiv, Kyiv, Ukraine)
E-mail: bohdanahladysh@gmail.com

Alexandr Prishlyak
(Taras Shevchenko National University of Kyiv, Kyiv, Ukraine)
E-mail: prishlyak@yahoo.com

Smooth functions are the tool of investigation in many scientific fields. Thus, their classification are important enough. There is a number of papers devoted to functions with non–degenerate singularities on a surface with the boundary [1–5]. Furthermore, there are significant results dedicated to topological structure of spaces of smooth functions with isolated critical points was presented in [6–10].

Let M (and N) be smooth compact connected oriented surface with the boundary ∂M. We consider the class of functions $\Omega(M) = \{f : M \to \mathbb{R} | CP(f) = NDCP(f) \supset CP(f|_{\partial M}) = NDCP(f|_{\partial M})\}$, where $CP(f)$ ($NDCP(f)$) is the set of (non-degenerated) critical points of f.

Theorem 1. *The following statements hold true:*

1) *for arbitrary function $f \in \Omega(M)$ there exists the m-function $g : M \to \mathbb{R}$ which is topologically equivalent to f;*
2) *for arbitrary m-function $g : M \to \mathbb{R}$ there exists the function $f \in \Omega(M)$ such that f and g are topologically equivalent.*

Definition 2. Smooth functions $f \in \Omega(M)$ and $g \in \Omega(N)$ are called $\mathcal{O}$*-equivalent* if there exists a homeomorphism $\lambda : M \to N$, which maps the components of the level sets of f onto the components of the level sets of g, preserve the growing directions of functions and preserve the orientation. The $\mathcal{O}$-equivalence class of the pair $(U, f|_U)$ is called and $\mathcal{O}$-atom for an oriented surface, where U is the union of connected component of the (small enough) neighborhood of critical level, which contain the critical points.

Let $f \in \Omega(M)$. The components of level lines of the function f are said to be layers. These layers are homeomorphic to the circle or to the line segment for regular values of the function. Then, the surface M can be considered as the union of layers and we get the foliation with singularities. We call a layer by the layer of the first (the second) type if it corresponds to the component homeomorphic to the line segment (circle). Let us consider the equivalence relation on M, such that points are equivalent if and only if they belong to the same layer. Thus, after examining of nature factor-topology we get the graph Γ_f with edges of two types. In such a way we get the classification of edges being called *edges division* of graph Γ_f.

Definition 3. The vertices with degrees 3 and 4 of graph Γ_f of function f, which are incident to the first type edges are said to be Y and X-*vertices* correspondingly.

Definition 4. *Equipped Reeb graph* of a function $f \in \Omega(M)$ is graph Γ_f with edges division, orientation and cycle order at Y and X-vertices.

Definition 5. Equipped Reeb graphs Γ_f and Γ_g of functions $f, g \in \Omega(M)$ are said to be *equivalent* by means of the isomorphism $\varphi : \Gamma_f \to \Gamma_g$ (denote by $\Gamma_f \sim \Gamma_g$ or $\Gamma_f \sim_\varphi \Gamma_g$) if φ satisfies the following statements:

(1) preserve the division of the edges;
(2) preserve the cycle orders of the edges at each X and Y-vertex;
(3) preserve the edges orientation.

Theorem 6. *Let M, N be smooth compact surfaces (with the boundaries), such that $f \in \Omega(M)$, $g \in \Omega(N)$. Then f and g are $\mathcal{O}$-equivalent if and only if their equipped Reeb graphs Γ_f and Γ_g are equivalent.*

Further the first and the second type edges of the graph Γ_f are said to be I and O-edges correspondingly. To define the genus of the surface we consider the following designation: let E_{I} (E_{O}) be a number of I-edges (O-edges) and V_{I} (V_{O}) be a number of the vertices which are incident only with I-edges (O-edges). The number of components of the boundary of the surface M we denote by ∂.

Theorem 7. *Let graph Γ_f of a function $f \in \Omega(M)$ includes either O-edges, or I-edges. Then the genus of the surface can be calculated from the formulas (1) and (2) correspondingly, where*

$$g_{\mathsf{O}} = E_{\mathsf{O}} - V_{\mathsf{O}} + 1 \tag{1}$$

$$g_{\mathsf{I}} = \frac{E_{\mathsf{I}} - V_{\mathsf{I}} + 2 - \partial}{2} \tag{2}$$

Definition 8. A vertex with degree 2 (3) of the graph Γ_f of a function f, which are incident with the edges of both types, are said to be T-*vertex* (D-*vertex*).

Theorem 9. *The genus of a surface can be calculated from the following formula*

$$g = g_{\mathsf{O}} + g_{\mathsf{I}} + V_{\mathsf{D}} + V_{\mathsf{T}} - c_{\mathsf{O}} - c_{\mathsf{I}} + 1 \tag{3}$$

where g_{O} is a summary genus of the subgraph which consists only edge of the second type, such that the genus of each graph components is defined by the formula 1, g_{I} is a summary genus of the subgraph which consists only edge of the first type, such that the genus of each graph components is defined by the formula 2, V_{D} is the number of D-vertices and V_{T} is the number of T-vertices, c_{O} is the number of connected components of the subgraph which consists only edge of the second type and c_{I} is the number of connected components of the subgraph which consists only edge of the first type.

Corollary 10. *Let f be m-function. Then the formulas 1, 2 and 3 hold true.*

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