A note on tensor product of Archimedean vector lattices

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Let E ,F and G be Archimedean vector lattices. We say that a linear operator $T : E \to F$ is a lattice homomorphism if $T(x \vee y) = Tx \vee Ty$ for every $x, y \in E$. A bilinear map $\Phi : E \times F \to G$ is said to be positive if $|\Phi(x, y)| \leq \Phi(|x|, |y|)$ for all $x \in E$ and $y \in F$. The bilinear map $\Phi : E \times F \to G$ is said to be lattice bilinear map (or lattice bimorphism) whenever it is separately lattice homomorphisms for each variable or equivalently, $|\Phi(x, y)| = \Phi(|x|, |y|)$ for all $x \in E$ and $y \in F$. Let E and F be Archimedean vector lattices. Then, by [6] and [7] there exists an Archimedean vector lattice $E\bar{\bigotimes}F$ (called Fremlin tensor product) and a map $\bigotimes : E \times F \to E\bar{\bigotimes}F$ with the following well-known properties :

- (i) $\bigotimes$ is a lattice bimorphism and represents $E \bigotimes F$ as a linear subspace of $E\bar{\bigotimes}F$.
- (ii) If G is any Archimedean vector lattice, there is a one to one correspondence between lattice bimorphisms $\psi : E \times F \to G$ and lattice homomorphisms $\tau : E\bar{\bigotimes}F \to G$ given by $\psi = \tau \bigotimes$.
- (iii) $E \bigotimes F$ is dense in $E\bar{\bigotimes}F$. It means that for any $u \in E \bigotimes F$ there exist $0 \leq x \in E$ and $0 \leq y \in F$ such that for every $\delta > 0$ there is a $v \in E \bigotimes F$ with $| u - v | \leq \delta x \otimes y$.
- (iv) $E \bigotimes F$ is order dense in $E\bar{\bigotimes}F$. That is, if $0 < u$ in $E\bar{\bigotimes}F$ there exist $0 < x$ in E, $0 < y$ in F such that $0 < x \otimes y \leq u$.
- (v) f $u \in E\bar{\bigotimes}F$ there exist $0 \leq x \in E$, $0 \leq y \in F$ such that $| u | \leq x \otimes y$.
- (vi) If G is any Archimedean vector lattice and $\Phi : E \times F \to G$ is a lattice bimorphism such that $\Phi(x, y) > 0$ whenever $x > 0$ in E and $y > 0$ in F, then $E\bar{\bigotimes}F$ may be identified with the lattice subspace of G generated by $\Phi(E \times F)$.

Definition 1. Let E be a vector lattice and U be a subset of E. U is called a solid subset if $| x | \leq | y |$ and $y \in U$ imply $x \in U$. A solid subspace of E is called an order ideal. An order closed ideal is said to be a band.

In this talk, we deal with tensor product of two order ideals. Let E be a uniformly complete vector lattice and $x \in E$. An order ideal generated by x is given by

$$U_x = \{y : | y | \leq \lambda | x | \quad for \quad some \quad \lambda > 0\}.$$

Order unit norm on U_x is given by

$$\|y\| = inf\{\lambda > 0 : | y | \leq \lambda | x |\}.$$

U_x is algebraically and order isomorphic to an AM space with unit. Every AM space with unit is a $C(K)$ space for some compact Hausdorff space K with unit. Hence, Fremlin tensor product of two order ideals generated by single elements is an order ideal. For general case, we need the definition of orthomorphism. A linear operator $T : E \to E$ is called an orthomorphism if it is both band preserving and order bounded. By using orthomorphism, we can show that Fremlin tensor product of two order ideals is an order ideal.

Theorem 2. *Fremlin tensor product of two order ideals generated by single elements in a uniformly complete vector lattice is an order ideal.*

For this subject, we give the following references.

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