Formal groups and algebraic cobordism

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We present our algorithm for constructing of Lazard's one dimensional universal commutative formal group. We apply it to some constructions of complex and algebraic cobordism.

Algorithm for constructing of Lazard's one dimensional universal formal group

Here we follow to [1, 5, 8]. We extract from Lazard [1] an algorithm of constructing of Lazard's one dimensional universal formal group law. The constructions of $n(n \geq 1)$-buds, formal groupoids and moduli spaces of one dimensional formal groups are investigated and are used. Let A_q and $A^{'}_q$ be the rings of polynomials $A_q = \mathbb{Z}[\alpha_1, \ldots, \alpha_q]$ and $A^{'}_q = \mathbb{Q}[\alpha_1, \ldots, \alpha_q]$.

Proposition 1. *The structure of the algorithm has the next form. For the (one-dimensional)* 1*-bud* $x + y + \alpha_1 xy$ *we put:*

$$f_1(x, y) = x + y + \alpha_1 xy;\ \varphi_1 = x - \frac{1}{2}\alpha_1 x^2;$$

$$f_{q+1}(x, y) = f_q(x, y) + h^{'}(x, y) + \alpha_{q+1} C_{q+2}(x, y),\ f_q(x, y) \in A_q;$$

compute $\varphi_{q+1},\ \varphi_{q+1} \in A^{'}_q$.

Remark 2. The algorithm and expressions for φ_{q+1}, $h^{'}(x, y)$, $C_{q+2}(x, y)$ will be given in my talk.

Definition 3. The ring $\mathbb{L} = \mathbb{Z}[\alpha_1, \alpha_2, \cdots]$ is called the Lazard ring.

Formal groups in complex cobordism

Here we follow to [2, 3, 4, 6].

Let M be a smooth manifold, TM be the tangent bundle on M and $\mathbb{R}^m$ the trivial real m-dimensional bundle of M.

Definition 4. A manifold M is *stably complex* if for some natural m the real vector bundle $TM \bigoplus \mathbb{R}^m$ admits a complex structure.

Let M_1 and M_2 be two smooth manifolds of dimension n, and W be the smooth manifold of dimension $n + 1$ with a boundary that is the union of M_1 and M_2, i.e. $\partial W = M_1 + M_2$.

Definition 5. Let M_1, M_2, W be stably complex manifolds. In above notations the *complex (unitary) cobordism* between M_1 and M_2 is a manifold W whose boundary is the disjoint union of M_1, M_2, $\partial W = M_1 + \overline{M}_2$ where the corresponding structure on ∂W is induced from W and $\overline{M}_2$ denotes the manifold with opposite structure.

Remark 6. Suppose we have the relation of complex cobordism. Then the relation divides stably complex manifolds on equivalence classes called classes of complex cobordisms.

Lemma 7. *The set of classes of complex cobordisms with operations of disjoint union and product of stable manifolds form commutative graded ring* $\mathbb{L} = \mathbb{Z}[v_1, v_2, \cdots]$.

Theorem 8 (A. S. Mishchenko). *Let $g(t)$ be the logarithm of Lazard universal formal group, and $[\mathbb{C}P^n]$ are classes of unitary cobordisms of complex projective spaces. Then*

$$g(t) = \sum_{n \geq 0} \frac{[\mathbb{C}P^n]}{n+1} t^{n+1}, \ [\mathbb{C}P^0] = 1.$$

Formal groups in algebraic cobordism

Here we follow to [7]. Let k be a field of characteristic zero and $\mathbf{Sm}(k)$ be the full subcategory of smooth quasi-projective k-schemes of the category of separable finite-type k-schemes. Let A^* be an oriented cohomology theory on $\mathbf{Sm}(k)$ and let $c_1(L)$ be the first Chern class of line bundle L on $X \in \mathbf{Sm}(k)$.

Proposition 9 (Quillen, [7]). *Let L, M be line bundles on $X \in \mathbf{Sm}(k)$. There exists formal group law F_A with coefficients in A^* such that*

$$c_1(L \otimes M) = F_A(c_1(L), c_1(M)).$$

Theorem 10 (Levine-Morel). *There is a universal oriented cohomology theory Ω over k called algebraic cobordism. The classifying map $\phi_\Omega : \mathbb{L} \to \Omega^*(k)$ is an isomorphism , so F_Ω is the universal formal group law.*

Problem 11. It seems that very little is known about applying $n (n \geq 2)$-dimensional commutative formal groups to cobordism theory. For instance what is the application of a two-dimensional 1-bud ?

References

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