Inverse problem for tree of Stietjes strings

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Finite-dimensional spectral problems on an interval were considered in [1] (some recent results see in [4], [3] and applications in [2]). Finite-dimensional spectral problems on graphs occur in various fields of physics (see e.g. [5], [6] and [7]).

We consider a tree T rooted at a pendant vertex. All edges are directed away from the root. Each edge e_j of this tree is a Stieltjes string with point masses $m_k^j \quad (k=1,2,\ldots,n_j,\ j=1,2,...,q)$ and subintervals $l_k^j \quad (k=0,1,\ldots,n_j)$. The total length of e_j is $l_j=\sum\limits_{k=0}^{n_j} l_k^j$. We denote $\tilde{n}_j=n_j-1$ if $l_{n_j}^j=0$ and $\tilde{n}_j=n_j$ if $l_{n_j}^j>0$. The Dirichlet problem on this tree consists of the following equations.

For each edge:

$$\frac{u_k^j-u_{k+1}^j}{l_k^j}+\frac{u_k^j-u_{k-1}^j}{l_{k-1}^j}-m_k^j\lambda^2 u_k^j=0, \quad (k=1,2,\ldots,\tilde{n}_j,\ j=1,2,...,q). \tag{1}$$

For each interior vertex with incoming edge e_j and outgoing edges e_r we have

$$u_{\tilde{n}_j+1}^j=u_0^r, \tag{2}$$

and

$$\frac{u_{\tilde{n}_j+1}^j-u_{\tilde{n}_j}^j}{l_{\tilde{n}_j}^j}+\sum_r \frac{u_0^r-u_1^r}{l_0^r}=\begin{cases} 0, & \text{if } l_{n_j}^j>0, \\ -m_{n_j}^j\lambda^2 u_{n_j}^j, & \text{if } l_{n_j}^j=0. \end{cases} \tag{3}$$

For an edge e_j incident with a pendant vertex (except of the root) we have the Dirichlet boundary condition:

$$u_{n_j+1}^j=0. \tag{4}$$

If e_1 is the edge incident with the root then at the root we have

$$u_0^1=0. \tag{5}$$

The Neumann problem consists of equations (1)–(4) and

$$u_0^1=u_1^1. \tag{6}$$

at the root.

First of all we notice that interior vertices of degree 2 do not influence the results and we can assume absence of such vertices without losses of generality. Let P be a path in the tree T involving the maximum number of masses. Obviously it starts and finishes with pendant vertices. We denote the initial vertex of P by v_0 and choose it as the root of the tree. The enumeration of other vertices is arbitrary. We denote the edge incoming into a vertex v_i by e_i for all i. Then $P: v_0\to v_1\to v_{s_2}\to v_{s_3}\to ...\to v_{s_{r-1}}\to v_{s_r}$. Deleting v_0 and e_1 we obtain a new tree T' rooted at the vertex v_1.

Since the degree of v_1 is $d(v_1)>2$ we can divide our tree T' into subtrees T_1', T_2', ..., $T_{d(v_1)-1}'$ having v_1 as the only common vertex. (We say that T_1', T_2', ..., $T_{d(v_1)-1}'$ are *complementary subtrees* of T'.

Denote by $\phi_{N_{(v_0)}}(z)$ (where $z = \lambda^2$) the characteristic polynomial of problem (1)– (4), (6) on the tree T and by $\phi_{D_{(v_0)}}(z)$ the characteristic polynomial of problem (1)– (4), (5) on this tree. These polynomials are normalized such that

$$\frac{\phi_{D(v_0)}(0)}{\phi_{N(v_0)}(0)} = l_1 + \frac{1}{\sum\limits_{r=1}^{d(v_1)-1} \frac{\phi_{Nr(v_1)}(0)}{\phi_{Dr(v_1)}(0)}},$$

$\phi_{D,r(v_1)}(z)$ is the characteristic polynomial of the Dirichlet problem (1)– (4), (5) on T'_r and $\phi_{N,r(v_1)}(z)$ is the characteristic polynomial of the Neumann problem (1)– (4), (6) on T'_r and so on.

The inverse problem lies in recovering the spectral problem data $\{m_k^j\}_{k=1}^{n_j}$, $\{l_k^j\}_{k=0}^{n_j}$ using the spectra $\{\mu_k\}_{k=-n,\ k\neq 0}^{n}$ and $\{\nu_k\}_{k=-n,\ k\neq 0}^{n}$ of the Neumann and Dirichlet problems.

The following theorem gives sufficient conditions for existence of solution of such inverse problem.

Theorem 1. *Let $\{\mu_k\}_{k=-n,\ k\neq 0}^{n}$ and $\{\nu_k\}_{k=-n,\ k\neq 0}^{n}$ be symmetric ($\mu_{-k} = -\mu_k$, $\nu_{-k} = -\nu_k$) and monotonic sequences of real numbers which interlace:*

$$0 < (\mu_1)^2 < (\nu_1)^2 < ... < (\mu_n)^2 < (\nu_n)^2.$$

Let T be a metric tree of a prescribed form rooted at a pendant vertex v_0 with prescribed lengths of edges $l_j > 0$ ($j = 1, 2, ..., q$, q is the number of edges in T).

Then

1) there exist numbers $n_j \in \{0\} \cup \mathbb{N}$ ($j = 1, 2, ..., q$), sequences of positive numbers $\{m_k^j\}_{k=1}^{n_j}$ (point masses on the edge e_j, $j = 1, 2, ..., q$) and numbers $\{l_k^j\}_{k=0}^{n_j}$ ($l_k^j > 0$ for all $k = 0, 1, ..., n_j - 1$, $l_{n_j} \geq 0$ for all $j = 1, 2, ..., g$ such that $\sum\limits_{k=0}^{n_j} l_k^j = l_j$, $\sum\limits_{j=1}^{q} n_j = n$, the spectrum of Neumann problem (1)–(4), (6), coincides with $\{\mu_k\}_{k=-n,\ k\neq 0}^{n}$ and the spectrum of Dirichlet problem (1)–(4), (5) coincides with $\{\nu_k\}_{k=-n,\ k\neq 0}^{n}$;

2) the two spectra $\{\mu_k\}_{k=-n,\ k\neq 0}^{n}$ and $\{\nu_k\}_{k=-n,\ k\neq 0}^{n}$ and the length l_1 of the edge incident with the root uniquely determine the masses $\{m_k^1\}_{k=1}^{n_1}$ (point masses on the edge e_1) and lengths $\{l_k^1\}_{k=0}^{n_1}$ of the subintervals on this edge.

References

[1] F.R.Gantmakher and M.G.Krein. Oscillating matrices and kernels and vibrations of mechanical systems. *AMS Chelsea Publishing, Providence, RI*, 2002.

[2] A. F. Filimonov and A. D. Myshkis. On properties of large wave effect in classical problem bead string vibration. *J. Difference Equations and Applications*, 10(13-15) : 1171–1175, 2004.

[3] Che-Wei Tsao, Chun-Kong Law. The Stieltjes string and its associated nodal points. *Operators and Matrices*, 13(2): 363–373, 2019.

[4] S.J. Cox, M. Embree and J.M. Hokanson. One can hear the composition of a string: experiments with an inverse eigenvalue problem. *SIAM Rev.*, 54(1) : 157–178, 2012.

[5] A. Morassi, G. Gladwell. Dynamical Inverse Problems: Theory and Applications. *CISM Courses and Lectures 529*, 1–29, 2011.

[6] G. Gladwell. Inverse problems in vibration. *SIAM Rev.*, 2004.

[7] J.S. Maybee, J. Genin. Mechanical vibration trees. *J. Math. Anal. Appl.*, 45: 746–763, 1974.