Estimate of maximum of the products of inner radii of mutually non-overlapping domains

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Let $\mathbb{N}$, $\mathbb{R}$ be the sets of natural and real numbers, respectively, $\mathbb{C}$ be the complex plane, $\overline{\mathbb{C}} = \mathbb{C}\bigcup\{\infty\}$ be its one point compactification, $\mathbb{U}$ be the open unit disk in $\mathbb{C}$ and $\mathbb{R}^+ = (0, \infty)$. Let $r(B, a)$ be an inner radius of the domain $B \subset \overline{\mathbb{C}}$ relative to a point $a \in B$ [1–4]. The inner radius of the domain B is connected with Green's generalized function $g_B(z, a)$ of the domain B by the relations

$$g_B(z, a) = -\ln|z - a| + \ln r(B, a) + o(1), \quad z \to a,$$

$$g_B(z, \infty) = \ln|z| + \ln r(B, \infty) + o(1), \quad z \to \infty.$$

The system of points $A_n := \{a_k \in \mathbb{C}, k = \overline{1, n}\}$, $n \in \mathbb{N}$, $n \geqslant 2$, is called n-radial, if $|a_k| \in \mathbb{R}^+$ for $k = \overline{1, n}$ and

$$0 = \arg a_1 < \arg a_2 < \ldots < \arg a_n < 2\pi.$$

Consider the following extremal problem.

Problem. For all values of the parameter $\gamma \in \mathbb{R}^+$ to find estimate of the maximum of the functional

$$J_n(\gamma) = [r(B_0, 0)\, r(B_\infty, \infty)]^\gamma \prod_{k=1}^{n} r(B_k, a_k), \tag{1}$$

where $n \in \mathbb{N}$, $n \geqslant 2$, $a_0 = 0$, $A_n = \{a_k\}_{k=1}^n \in \overline{\mathbb{C}}/\{0, \infty\}$ be any fixed n-radial system of different points, B_0, B_∞, $\{B_k\}_{k=1}^n$ be a system of mutually non-overlapping domains, $0 \in B_0 \subset \overline{\mathbb{C}}$, $\infty \in B_\infty \subset \overline{\mathbb{C}}$, $a_k \in B_k \subset \overline{\mathbb{C}}$, $k = \overline{1, n}$.

The following proposition is true.

Theorem 1. *Let $n \in \mathbb{N}$, $n \geqslant 2$, $\gamma \in \mathbb{R}^+$. Then, for any fixed n-radial system of different points $A_n = \{a_k\}_{k=1}^n \in \overline{\mathbb{C}}/\{0, \infty\}$ and any mutually non-overlapping domains B_0, B_∞, B_k, $a_0 = 0 \in B_0 \subset \overline{\mathbb{C}}$, $\infty \in B_\infty \subset \overline{\mathbb{C}}$, $a_k \in B_k \subset \overline{\mathbb{C}}$, $k = \overline{1, n}$, the following inequalities hold*

$$J_n(\gamma) \leqslant \begin{cases} (n+1)^{-\gamma\frac{n+1}{n+2}} \left[\prod\limits_{k=1}^{n} r(B_k, a_k)\right]^{1-\frac{2\gamma}{n+2}} \prod\limits_{k=1}^{n} |a_k|^{\frac{2\gamma}{n+2}}, & \text{if} \quad \gamma \in \left(0, \frac{n+2}{2}\right]; \\ (n+1)^{-\frac{n+1}{2}} \prod\limits_{k=1}^{n} |a_k|, & \text{if} \quad \gamma > \frac{n+2}{2}. \end{cases}$$

Remark 2. If $\gamma = \frac{n+2}{2}$ and $\prod\limits_{k=1}^{n} |a_k| \leqslant 1$, then from above posed Theorem 1, the following inequality holds

$$[r(B_0, 0)\, r(B_\infty, \infty)]^{\frac{n+2}{2}} \prod_{k=1}^{n} r(B_k, a_k) \leqslant (n+1)^{-\frac{n+1}{2}}.$$

In this case the structure of points and domains is not important.

For any n-radial system of points $A_n = \{a_k\}_{k=1}^n$, $|a_k| = 1$, and any pairwise non-overlapping domains $\{B_k\}_{k=1}^n$, $a_k \in B_k \subset \overline{\mathbb{C}}$, $k = \overline{1,n}$, the inequality

$$\prod_{k=1}^{n} r(B_k, a_k) \leqslant 2^n \prod_{k=1}^{n} \alpha_k$$

is valid (see Corollary 5.1.3 [1]). In Theorem 6.11 [2] for any different points a_k on the circle $|a_k| = 1$, $k = \overline{1,n}$ $(n \geqslant 2)$, and any pairwise non-overlapping domains $B_k \subset \overline{\mathbb{C}}$ such that $a_k \in B_k$, $k = \overline{1,n}$, the inequality

$$\prod_{k=1}^{n} r\left(B_k, a_k\right) \leqslant \left(\frac{4}{n}\right)^n$$

is proved. Thus, from Theorem 1 we have next result.

Corollary 3. *Let $n \in \mathbb{N}$, $n \geqslant 2$, $\gamma \in \mathbb{R}^+$. Then, for any system of different points $\{a_k\}_{k=1}^n$ of the unit circle $|a_k| = 1$ and any mutually non-overlapping domains B_0, B_∞, B_k, $a_0 = 0 \in B_0 \subset \overline{\mathbb{C}}$, $\infty \in B_\infty \subset \overline{\mathbb{C}}$, $a_k \in B_k \subset \overline{\mathbb{C}}$, $k = \overline{1,n}$, the following inequalities hold*

$$J_n(\gamma) \leqslant \begin{cases} (n+1)^{-\gamma\frac{n+1}{n+2}} \left(\frac{4}{n}\right)^{n-\frac{2\gamma n}{n+2}}, & \text{if} \quad \gamma \in \left(0, \frac{n+2}{2}\right]; \\ \\ (n+1)^{-\frac{n+1}{2}}, & \text{if} \quad \gamma > \frac{n+2}{2}. \end{cases}$$

If $B_0 \subset \mathbb{U}$, then from the proof of the Theorem 1, the following results are valid.

Corollary 4. *Let $n \in \mathbb{N}$, $n \geqslant 2$, $\gamma \in \mathbb{R}^+$ and $B_0 \subset \mathbb{U}$. Then, for any system of different points $\{a_k\}_{k=1}^n$ of the unit circle $|z| = 1$ and any mutually non-overlapping domains B_k, $a_k \in B_k \subset \overline{\mathbb{C}}$, $k = \overline{0,n}$, $a_0 = 0$, and the domains B_k, $k = \overline{1,n}$, are mirror-symmetric relative to the unit circle $|z| = 1$, the inequality holds*

$$r^{2\gamma}\left(B_0, 0\right) \prod_{k=1}^{n} r\left(B_k, a_k\right) \leqslant (n+1)^{-\frac{n+1}{2}}.$$

Corollary 5. *Let $n \in \mathbb{N}$, $n \geqslant 2$, $\gamma \in \mathbb{R}^+$, $R > 0$ and B_0 be an arbitrary domain belonging to the open circle $|w| < R$. Then, for any n-radial system of different points $A_n = \{a_k\}_{k=1}^n$ such that $|a_k| = R$, $k = \overline{1,n}$, and any mutually non-overlapping domains B_k, $a_k \in B_k \subset \overline{\mathbb{C}}$, $k = \overline{0,n}$, $a_0 = 0$, and the domains B_k, $k = \overline{1,n}$, are mirror-symmetric relative to the circle $|w| = R$, the inequality holds*

$$r^{2\gamma}\left(B_0, 0\right) \prod_{k=1}^{n} r\left(B_k, a_k\right) \leqslant (n+1)^{-\frac{n+1}{2}} \cdot R^{2\gamma}.$$

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