Uniqueness of pretangent spaces at infinity

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The pretangent spaces to an unbounded metric space (X, d) at infinity are, by definition, some limits of rescaling metric spaces $\left(X, \frac{1}{r_n}d\right)$ with r_n tending to infinity. The Gromov–Hausdorff convergence and the asymptotic cones are most often used for construction of such limits. Both of these constructions are based on high–order logic abstractions (see, e.g., [5]), which makes them very powerful, but it does away the constructiveness. In [2, 3] we present and consider a more constructive approach to building an asymptotic structure of unbounded metric spaces at infinity.

The object of the present abstract is metric spaces having with unique pretangent spaces at infinity. The theorem below gives the necessary and sufficient conditions under which an unbounded metric space X has an unique pretangent space $\Omega^X_{\infty,\tilde{r}}$ at infinity for every fixed scaling sequence $\tilde{r}$.

We will denote by $\mathfrak{A}$ the class of spaces having the property

$$((X, d) \in \mathfrak{A}) \Leftrightarrow ((X, d) \text{ is unbounded and } \forall\ \tilde{r} \text{ there is a unique } \Omega^X_{\infty,\tilde{r}}).$$

Let (X, d) be a metric space and let $p \in X$. For each pair of nonempty sets C, $D \subseteq X$ write

$$\Delta(C, D) := \sup\{d(x, y)\colon x \in C, y \in D\}.$$

In addition we define, for every $\varepsilon \in (0, 1)$, the set S^2_ε as

$$S^2_\varepsilon := \left\{(r, t) \in Sp^2(X)\colon r \neq 0 \neq t \text{ and } \left|\frac{r}{t} - 1\right| \geq \varepsilon\right\},$$

where $Sp^2(X)$ is the Cartesian square of $Sp(X) = \{d(x, p)\colon x \in X\}$.

Theorem 1. [4] *Let (X, d) be an unbounded metric space and let p be a point of X. Then $(X, d) \in \mathfrak{A}$ if and only if the following conditions are satisfied simultaneously.*

(1) *The limit relations*

$$\lim_{k\to 1} \limsup_{r\to\infty} \frac{\operatorname{diam}(A(p, r, k))}{r} = \lim_{r\to\infty} \frac{\operatorname{diam}(S(p, r))}{r} = 0$$

hold, where $r \in (0, \infty)$, $k \in [1, \infty)$, $A(p, r, k)$ is the annulus $\{x \in X\colon \frac{r}{k} \leq d(x, p) \leq rk\}$ and $S(p, r)$ is the sphere $\{x \in X\colon d(x, p) = r\}$.

(2) *Let $\varepsilon \in (0, 1)$. If $((q_n, t_n))_{n\in\mathbb{N}} \subset S^2_\varepsilon$ and*

$$\lim_{n\to\infty} q_n = \lim_{n\to\infty} t_n = \infty,$$

and there is

$$\lim_{n\to\infty} \frac{q_n}{t_n} = c_0 \in [0, \infty],$$

then there exists a finite limit

$$\lim_{n\to\infty} \frac{\Delta(S(p, q_n), S(p, t_n))}{|q_n - t_n|} := \kappa_0.$$

It can be proved that conditions (1) and (2) from Theorem 1 are mutually independent in the sense that no one of them implies the another.

Corollary 2. *Let (X,d) be a metric space and let Y be an unbounded subspace X. Then $(X,d) \in \mathfrak{A}$ implies $(Y,d) \in \mathfrak{A}$.*

Consider, for simplicity, logarithmic spirals having the pole at 0. The polar equation of these spirals is

$$\rho = kb^{\varphi}, \tag{1}$$

where k and b are constants, $k \in (0,\infty)$ and $b \in (0,1) \cup (1,\infty)$. The rotation of the polar axis on the angle $\varphi_1 = -\frac{\ln k}{\ln b}$ transforms (1) to the form

$$\rho = b^{\varphi}. \tag{2}$$

Let us denote by $\mathbb{S}^*$ the set of all complex numbers lying on spiral (2) and let

$$\mathbb{S} = \mathbb{S}^* \cup \{0\},$$

i.e., $\mathbb{S}$ is the closure of $\mathbb{S}^*$ in the complex plane $\mathbb{C}$. In the following theorem we consider $\mathbb{S}$ as a metric space with the usual metric $d(z,w) = |z-w|$.

Theorem 3. [4] *Each pretangent space to $(\mathbb{S},d)$ at infinity is unique and isometric to $\mathbb{S}$.*

These results have natural infinitesimal analogs (see [1]).

References

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