Hyperbolic 4-cobordisms, Teichmuller spaces and quasiregular mappings in space

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We present a new effect in the theory of deformations of hyperbolic 3-manifolds/orbifolds or their uniform hyperbolic lattices $\Gamma \subset \operatorname{Isom} H^3$ (i.e. in the Teichmüller spaces of conformally flat structures on closed hyperbolic 3-manifolds, cf. [1, 2]). We show that such Teichmüller space or the corresponding variety of conjugacy classes of discrete representations $\rho : \Gamma \to \operatorname{Isom} H^4$ may have connected components whose dimensions differ by arbitrary large numbers, cf. [3, 5]. This is based on our "Siamese twins construction" of non-faithful discrete representations of hyperbolic lattices related to non-trivial "symmetric hyperbolic 4-cobordisms" [8] and Gromov-Piatetski-Shapiro interbreeding construction [11]. There are several applications of this result, from new non-trivial hyperbolic homology 4-cobordisms (cf. [9]) and wild 2-knots in the 4-sphere (cf. [5]), to bounded quasiregular locally homeomorphic mappings in 3-space, especially to their asymptotics in the unit 3-ball and to quasisymmetric embeddings of a closed ball inextensible in neighborhoods of any boundary points (cf. [10, 12, 13, 14, 15]).

The last part is based on our construction of a new type of bounded locally homeomorphic quasiregular mappings in 3-sphere (and in the unit 3-ball), see [6]. It addresses long standing problems for such mappings, including M. A. Lavrentiev problem, Pierre Fatou problem and Matti Vuorinen injectivity and asymptotics problems (cf. [7]). The construction of such mappings comes from our construction of non-trivial compact 4-dimensional cobordisms M with symmetric boundary components and whose interiors have complete 4-dimensional real hyperbolic structures (cf. [4]). Such bounded locally homeomorphic quasiregular mappings are defined in the unit 3-ball $B^3 \subset \mathbb{R}^3$ as mappings equivariant with the standard conformal action of uniform hyperbolic lattices $\Gamma \subset \operatorname{Isom} H^3$ in the unit 3-ball and with its discrete representation $G = \rho(\Gamma) \subset \operatorname{Isom} H^4$ (cf. [6]). Here G is the fundamental group of our non-trivial hyperbolic 4-cobordism $M = (H^4 \cup \Omega(G))/G$ and the kernel of the homomorphism $\rho : \Gamma \to G$ is a free group F_m on arbitrary large number m generators.

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