On the geometry of submersions

G. M. Abdishukurova
(National University of Uzbekistan, Tashkent, 100174, Tashkent, Uzbekistan)
E-mail: Abdishukurova93@yandex.ru

A. Ya. Narmanov
(National University of Uzbekistan, Tashkent, 100174, Tashkent, Uzbekistan)
E-mail: narmanov@yandex.ru

Let M be a smooth connected Riemannian manifold of dimension n with Riemannian metric g.

Definition 1. Differentiable mapping $\pi : M \to B$ of maximal rank, where B is a smooth Riemannian manifold of dimension m, called submersion for $n > m$.

Submersion of $\pi : M \to B$ generates a foliation F of dimension $k = n - m$ on the manifold M, whose leaves are the submanifolds $L_p = \pi^{-1}(p), p \in B$. For a point $q \in L_p$ we denote by T_qF the tangent space of the leaf L_p at the point q, by $H(q)$ the orthogonal complement of the tangent space T_qF of the leaf L_p, i.e. $T_qM = T_qF \oplus H(q)$. We have two distributions $TF : q \to T_qF$, $H : q \to H(q)$. Each vector field X can be represented as $X = X^v + X^h$, where X^v, X^h are the orthogonal projections of X onto TF, H respectively. Here, for convenience, TF, H are considered as subbundles of the tangent bundle TM. If $X^h = 0$, then X is called a vertical field (it is tangent to the foliation), and if $X^v = 0$, then X is called a horizontal field.

Definition 2. A diffeomorphism $\varphi : M \to M$ is called a diffeomorphism of the foliated manifold (M, F), if the image $\varphi(L_\alpha)$ of each leaf L_α is a leaf of the foliation F.

The diffeomorphism $\varphi : M \to M$ of the foliated manifold (M, F), is denoted by $\varphi : (M, F) \to (M, F)$. The set of all diffeomorphisms of a foliated manifold is denoted by $Diff_F(M)$. The set $Diff_F(M)$ is a group with respect to the superposition of mappings and is a subgroup of the group $Diff(M)$ of diffeomorphisms of the manifold M. The group $Diff_F(M)$ was studied in [2], in particular, it was proved that this group is a closed subgroup of the group $Diff(M)$ with respect to a compactly open topology.

Definition 3. A diffeomorphism $\varphi : (M, F) \to (M, F)$ is called an isometry of the foliated manifold (M, F), if the restriction of the mapping φ to each leaf of the foliation F is an isometry, that is, for each leaf L_α the map $\varphi : L_\alpha \to f(L_\alpha)$ is an isometry between the manifolds L_α and $\varphi(L_\alpha)$.

Denote by $G_F(M)$ the set of isometries of the foliated manifold (M, F). The group $G_F(M)$ is subgroup of $Diff(M)$ and therefore it is topological group in compact open topology.

Let us consider submersion $\pi : R^{n+1} \to R^1$, where

$$\pi(x_1, x_2, \cdots, x_n, x_{n+1}) = x_{n+1} - f(x_1, x_2, \cdots, x_n), \tag{1}$$

$f(x_1, x_2, \cdots, x_n)$ is a differentiable function.

Theorem 4. *Diffeomorphism $\varphi : R^{n+1} \to R^{n+1}$, defined by formula*

$$\varphi_\lambda(x_1, x_2, \cdots x_n, x_{n+1}) = (x_1, x_2, \cdots, x_n, x_{n+1} + \lambda\pi) \tag{2}$$

at $\lambda \neq -1$ is an isometry of foliation, generated by submersion (1).

Theorem 5. *The set of diffeomorphisms*

$$G_\Lambda = \{\varphi_\lambda : \lambda \in R^1, \lambda \neq -1\}, \tag{3}$$

is a subgroup of the group $G_F(M)$.

Using the mapping $\varphi_\lambda \to \lambda$ we identify the set G_Λ with the set $R^1 \setminus \{-1\}$ of real numbers other than -1. On the set $R^1 \setminus \{-1\}$ we define the multiplication as follows

$$\lambda_1 \cdot \lambda_2 \to \lambda_1 + \lambda_2 + \lambda_1\lambda_2, \tag{4}$$

The inverse element is determined by the formula

$$\lambda \to -\frac{\lambda}{1+\lambda} \tag{5}$$

and it is obvious that they are differentiable. Therefore, we have following.

Proposition 6. *The set G_Λ is a one-dimensional Lie group.*

Example 7. Consider the submersion $\pi : R^3 \to R^1$, where $\pi(x_1, x_2, x_3) = x_3 - f(x_1, x_2)$, $f(x_1, x_2) = x_1^2 + x_2^2$. This submersion generates a two-dimensional foliation F. The following vector fields

$$V_1 = \frac{\partial}{\partial x_1} + 2x_1\frac{\partial}{\partial x_3}, V_2 = \frac{\partial}{\partial x_2} + 2x_2\frac{\partial}{\partial x_3}$$

are vertical vector fields. Vector field

$$X = -x_2\frac{\partial}{\partial x_1} + x_1\frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_3}$$

is a foliated vector field for the foliation F, as shown by the following equalities $[V_1, X] = V_2, [V_2, X] = -V_1$. It is known that the flow of foliated vector field consists of diffeomorphisms of foliated manifold (M, F) [1]. The vector field X is a Killing vector field.Therefore, the flow of a vector field X consists of isometries of a foliated manifold. Indeed, the flow of the vector field X consists of diffeomorphisms

$$x \to A(t)x + bt$$

where $t \in R$, $b = \{0, 0, 1\}^T$, $x = (x_1, x_2, x_3)^T$,

$$A(t) = \begin{pmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

which are isometries of the foliated manifold (F, R^3).

Theorem 8. *Suppose for a vector field*

$$V = \sum_{i=1}^{n} \xi_i \frac{\partial}{\partial x_i}$$

holds equaality $V(f) = 0$. Then the flow of the vector field

$$X = V + \frac{\partial}{\partial x_{n+1}}$$

consists of diffeomorphisms of the foliated manifold (F, R^{n+1}) generated by submersion (1). If the field V is a Killing field, then the flow of the vector field X consists of isometries of the foliated manifold (F, R^{n+1}).

References

[1] Molino P. Riemannian foliations. Boston–Basel: Burkhauser, 1988.
[2] Narmanov A.Y., Zoyidov A.N. On the group of diffeomorphisms of foliated manifolds. Vestnik Udmurtskogo Universiteta. Matematika. Mekhanika. Kompyuternye Nauki, 2020, vol. 30, issue 1, pp. 49–58.