Semi-lattice of varieties of quasigroups with linearity

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We consider two-placed functions defined on an arbitrary set Q, called *carrier*, whose domain is $Q \times Q$ and the functions are called *binary operations. Left* and *right multiplications* are defined on the set of all binary operations $\mathcal{O}_2$ by

$$(f \underset{\ell}{\oplus} g)(x,y) := f(g(x,y),y) \quad \text{and} \quad (f \underset{r}{\oplus} g)(x,y) := f(x, g(x,y))$$

respectively. The obtained groupoids $(\mathcal{O}_2; \underset{\ell}{\oplus})$ and $(\mathcal{O}_2; \underset{r}{\oplus})$ are called *left* and *right symmetric monoids.* An operation f is called *left (right) invertible* if it has an invertible element ${}^{\ell}f$ $({}^{r}f)$ in the left (right) symmetric monoid and *invertible* if both are true. ${}^{\ell}f$ and ${}^{r}f$ are called *left* and *right divisions.* The sequence f, ${}^{\ell}f$, ${}^{r}({}^{\ell}f)$, ${}^{r}f$, $\ldots$ consists of at most six different operations called parastrophes of f:

$${}^{\sigma}f(x_{1\sigma}, x_{2\sigma}) = x_{3\sigma} :\Leftrightarrow\ f(x_1, x_2) = x_3, \quad \sigma \in S_3 := \{\tau \mid \tau \text{ is a permutation of } \{1,2,3\}\}.$$

Every parastrophe of an invertible operation is invertible. In other words, the set of all invertible binary operations Δ_2 defined on a carrier Q is parastrophically closed. The group $\mathrm{Ps}(f) := \{\tau \mid {}^{\tau}f = f\}$ is a subgroup of S_3 and called *parastrophic symmetry group* of f. $(Q; f, {}^{\ell}f, {}^{r}f)$ is *quasigroup*, if f is invertible operation and Q is its carrier. If, in addition, the operation f is associative, then the quasigroup is a group, i.e., it has a neutral element and every element has an inverse.

Two operations f and g are called *isotopic* if there exists a triple of bijections (δ, ν, γ) called an *isotopism* such that $f(x,y) := \gamma g(\delta^{-1}x, \nu^{-1}y)$ for all $x, y \in Q$. An isotope of a group is called a *group isotope.* A class of quasigroups is called a variety if it described by identities.

In most cases solving a problem for some set Φ of invertible functions, we solve it for all parastrophes of functions from Φ. That is why Φ is suppose to be parastrophically closed. Very often it is difficult or even impossible to find the general solution of a problem for all operations from Φ.

Partition problem: *Find a partition of the given parastrophically closed set of invertible functions into parastrophically closed subsets with respect to the given property.*

The second author [1] solved this problem for group isotopes with respect to parastrophic symmetry group. Here we consider a partition of group isotopes on an arbitrary carrier with respect to the property of linearity. Let $(Q; \cdot)$ be a group isotope and 0 be an arbitrary element of Q, then the right part of the formula $x \cdot y = \alpha x + a + \beta y$ is called a 0-*canonical decomposition*, if $(Q; +, 0)$ is a group and $\alpha 0 = \beta 0 = 0$. An arbitrary element b uniquely defines b-canonical decomposition of an arbitrary group isotope [2].

In this case: the element 0 is a *defining element*; $(Q; +)$ is a *decomposition group*; a is its *free member*; α is its *left*, i.e., *2-coefficient*; β is its *right*, i.e., *1-coefficient*; $J\alpha\beta^{-1}$ is its *middle*, i.e., *3-coefficient.*

An isotope of an arbitrary group with a canonical decomposition $x \cdot y = \alpha x + a + \beta y$ is called

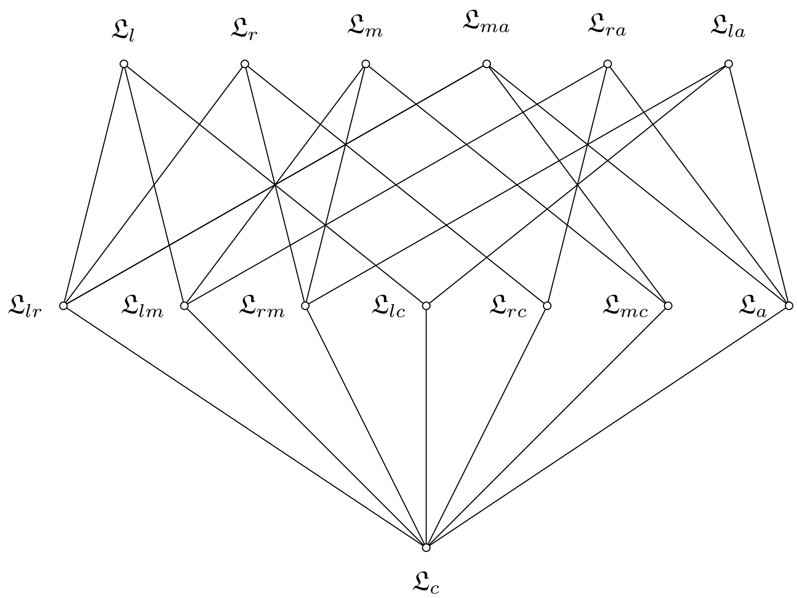
- *(strictly) i-linear*, if the i-coefficient is an automorphism of the decomposition group, where $i = 1, 2, 3$ (another coefficients are neither automorphism nor anti-automorphism);
- *(strictly) i-alinear*, if the i-coefficient is an anti-automorphism of the decomposition group, where $i = 1, 2, 3$ (another coefficients are neither automorphism nor anti-automorphism);
- *semi-linear or one-sided linear*, if it is i-linear for some $i = 1, 2, 3$;

- *semi-central or one-sided central*, if it is semi-linear and its decomposition group is commutative;
- *central* if it is linear and its decomposition group is commutative;
- *semi-alinear or one-sided alinear*, if it is i-alinear for some $i = 1, 2, 3$;
- *ij-linear*, if it is i-linear and j-linear; *linear*, if it is i-linear and j-linear for some $i \neq j$;
- *alinear*, if it is i-alinear for all $i, j \in \{1, 2, 3\}$.
- *anti-linear*, if all its coefficients are neither automorphism nor anti-automorphism.

Theorem 1. *All group isotope operations on an arbitrary carrier is parted into the following parastrophically closed blocks: 1) strictly semi-linear operations whose decomposition groups are not commutative; 2) strictly semi-alinear operations whose decomposition groups are not commutative; 3) linear operations whose decomposition groups are not commutative; 4) alinear operations whose decomposition groups are not commutative; 5) strictly semi-central operations; 6) central operations; 7) anti-linear operations.*

Theorem 2. *Let $i \in \{1, 2, 3\}$ and $\sigma \in S_3$. If a group isotope is i-linear (i-alinear), then its σ-parastrophe is $i\sigma^{-1}$-linear ($i\sigma^{-1}$-alinear).*

Quasigroups with linearity form 14 varieties: 1) three pairwise parastrophic vatieties of one-sided linear quasigroups: left $\mathfrak{L}_\ell$, right $\mathfrak{L}_r$ and middle $\mathfrak{L}_m$; 2) three pairwise parastrophic vatieties of one-sided alinear quasigroups: left $\mathfrak{L}_{a\ell}$, right $\mathfrak{L}_{ar}$ and middle $\mathfrak{L}_{am}$; 3) three pairwise parastrophic vatieties of linear quasigroups: left-right linear $\mathfrak{L}_{\ell r}$ (consequently they are middle alinear), left-middle linear $\mathfrak{L}_{\ell m}$ (consequently they are right alinear) and right-middle $\mathfrak{L}_{rm}$ linear (consequently they are left alinear); 4) a variety consisting of all left-right-middle alinear quasigroups $\mathfrak{L}_a$; 5) three pairwise parastrophic vatieties of one-sided central quasigroups: left $\mathfrak{L}_{\ell c}$, right $\mathfrak{L}_{rc}$ and middle $\mathfrak{L}_{mc}$; and 6) the variety of all central quasigroup $\mathfrak{L}_c$. The varieties form the following semi-latice.



References

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[2] F.N. Sokhatskii. About group isotopes I. *Ukrainian Math. J.*, 47(10): 1585–1598, 1995.