On the relationship between the Smith normal forms of matrices and of their least common multiple

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Let R be a commutative principal ideal domain with $1 \neq 0$, $M_n(R)$ be a ring $n \times n$ matrices over R and let $A, B \in M_n(R)$.

If $A = BC$, then we will say that B is a left divisor of matrix A and A is a right multiple of B. Moreover, if $M = AA_1 = BB_1$ then the matrix M is called a common right multiple of matrices A and B. If in addition the matrix M is a left divisor of any other common right multiple of matrices A and B then we say that M is a **least common right multiple** of A and B. ($[A, B]_r$ in notation).

In the 80's of the last century, M. Newman formulated the problem to establish of the relationship between the Smith normal forms of matrices and of their least common multiple and greatest common divisor over commutative principal ideal domain. The most important results in the study of M. Newman's problem are obtained in the papers [1], [2, 3] and [4].

Let A be non-singular $n \times n$ matrix over R. Then there exist invertible matrices P_A, Q_A , such that

$$P_A A Q_A = \mathrm{E} = \mathrm{diag}(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_\mathrm{n}), \ \text{ where } \ \varepsilon_\mathrm{i} \,|\varepsilon_{\mathrm{i}+1} \ , \ \mathrm{i} = 1, \ldots \mathrm{n} - 1.$$

The matrix E is called the Smith normal form, P_A and Q_A are called left and right transforming matrices for matrix A. Denote by $\mathbf{P}_A$ the set of all left transforming matrices for matrix A. Let the Smith normal form of the matrix B is

$$B \sim \Delta = diag(\delta_1, \delta_2, \ldots, \delta_n), \ \ where \ \ \delta_i \,|\delta_{i+1} \ , \ i = 1, \ldots n - 1.$$

Consider the set of matrices

$$\mathbf{L}(\mathrm{E}, \Delta) = \{\mathrm{L} \in \mathrm{GL_n(R)} \ \ | \ \ \exists \mathrm{L}_1 \in \mathrm{M_n(R)} : \mathrm{LE} = \Delta \mathrm{L}_1\}.$$

Theorem 1. *Let R be a commutative principal ideal domain and let*

$$A \sim diag(1, \varepsilon, \ldots, \varepsilon), \ B \sim diag(1, \ldots, 1, \delta),$$

$P_B P_A^{-1} = \|s_{ij}\|_1^n, \ P_B \in \mathbf{P}_B, \ P_A \in \mathbf{P}_A.$ *Then*

$$[A, B]_r = (L_A P_A)^{-1} \Omega = (L_B P_B)^{-1} \Omega,$$

where

$$\Omega = diag\Big(\frac{(\varepsilon, \delta)}{((\varepsilon, \delta), s_{n1})}, \varepsilon, \ldots, \varepsilon, [\varepsilon, \delta]\Big),$$

L_A , L_B *belong to sets* $\mathbf{L}(\Omega, \mathrm{E})$, $\mathbf{L}(\Omega, \Delta)$ *respectively and satisfy the equality:*

$$(P_B P_A^{-1}) L_A = L_B.$$

References

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