Inversion with respect to an elliptic cycle of a hyperbolic plane of positive curvature

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In the Cayley – Klein projective interpretation, the hyperbolic plane $\widehat{H}$ of positive curvature (see, for instance, [1], [2]) is defined as the exterior (with respect to the oval curve γ, called the absolute of the plane $\widehat{H}$) domain of the projective plane P_2. A complete Lobachevskii plane Λ^2 is realized on the domain of the plane P_2 that is interior with respect to the oval curve γ. The planes Λ^2 and $\widehat{H}$ are components of the expanded hyperbolic plane H^2. The group G of projective automorphisms of the oval curve γ is the fundamental group of transformations for $\widehat{H}$, H^2, and the Lobachevskii plane Λ^2.

In articles [3] and [4], we studied the inversion of the plane $\widehat{H}$ with respect to a hypercycle and a horocycle, respectively. In the present work we investigate the inversion $\mathbf{I}$ with respect to an elliptic cycle of the plane $\widehat{H}$. We obtain the analitical expression of the inversion in the canonical frame of the first type, and we define the images of lines and cycles which are concentric with the inversion base. Also we investigate the horizons of elliptic and hyperbolic cycles of the plane $\widehat{H}$.

We formulate the main results of our research in the following theorems. We denote the cycle of the plane $\widehat{H}$ with centre S and radius r by $\omega(S, r)$.

Theorem 1. *Let $\omega(S, r)$ be an elliptic cycle of the hyperbolic plane $\widehat{H}$ of positive curvature. Assume that a hyperbolic cycle $\omega_0(S, r_0)$ of the plane $\widehat{H}$ has the common horizon $\widetilde{\omega}(S, \widetilde{r})$ with $\omega(S, r)$. Then the inversion $\boldsymbol{I}$ with respect to the elliptic cycle $\omega(S, r)$ possesses the following properties.*

1. The cycles $\omega(S, r)$ and $\omega_0(S, r_0)$ are invariant under the inversion $\boldsymbol{I}$. The cycle $\widetilde{\omega}(S, \widetilde{r})$ corresponds under $\boldsymbol{I}$ to the absolute of the plane $\widehat{H}$.

2. An elliptic cycle $\omega_e(S, r_e)$ of $\widehat{H}$ corresponds under $\boldsymbol{I}$ to an elliptic cycle $\omega'_e(S, r'_e)$ with radius r'_e which satisfies the following conditions:

$$\operatorname{tg}\frac{r'_e}{\rho} = \operatorname{tg}^2\frac{r}{\rho}\operatorname{ctg}\frac{r_e}{\rho}, \quad \operatorname{tg}\frac{r'_e}{\rho} = \tanh\frac{\widetilde{r}}{\rho}\operatorname{ctg}\frac{r_e}{\rho}, \quad \operatorname{tg}\frac{r'_e}{\rho} = \tanh^2\frac{r_0}{\rho}\operatorname{ctg}\frac{r_e}{\rho},$$

where ρ, $\rho \in \mathbb{R}_+$, is a curvature radius of the plane $\widetilde{H}$.

The cycles $\omega_e(S, r_e)$ and $\omega'_e(S, r'_e)$ lie in the different domains of the point S valiana with respect to the cycle $\omega(S, r)$. If $r = \pi\rho/4$, then the height of the cycle $\omega'_e(S, r'_e)$ is equal to r_e.

3. Assume that $r < \pi\rho/4$, that is, the cycles $\widetilde{\omega}(S, \widetilde{r})$ and $\omega_0(S, r_0)$ lie in the plane $\widehat{H}$.

If a hyperbolic cycle $\omega_h(S, r_h)$ of the plane H^2 is ixterior with respect to the cycle $\widetilde{\omega}(S, \widetilde{r})$, then under $\boldsymbol{I}$ the cycle $\omega_h(S, r_h)$ corresponds to an equidistant $\omega'_h(S, r'_h)$ of the plane Λ^2. In this case the module of the height of the equidistant $\omega'_h(S, r'_h)$ grows when the radius of the cycle $\omega_h(S, r_h)$ grows.

If a hyperbolic cycle $\omega_h(S, r_h)$ of the plane H^2 lies between the cycles $\widetilde{\omega}(S, \widetilde{r})$ and $\omega_0(S, r_0)$, then under $\boldsymbol{I}$ the cycle $\omega_h(S, r_h)$ corresponds to a hyperbolic cycle $\omega'_h(S, r'_h)$ between the absolute of the plane H and the cycle $\omega_0(S, r_0)$. In this case the cycles $\omega_h(S, r_h)$ and $\omega'_h(S, r'_h)$ together approach the cycle ω_0 or move away from this cycle.

4. Assume that $r = \pi\rho/4$, that is, the cycles $\widetilde{\omega}$ and ω_0 coincide with the absolute of the plane $\widehat{H}$. The hyperbolic cycle $\omega_h\,(S, r_h)$ of the plane $\widehat{H}$ corresponds under the inversion $\boldsymbol{I}$ to an equidistant $\omega'_h\,(S, r'_h)$ of the Lobachevskii plane. In this case the module of the height of the equidistant $\omega'_h(S, r'_h)$ grows when the radius of the cycle $\omega_h(S, r_h)$ grows.

5. Assume that $r > \pi\rho/4$, that is, the cycles $\widetilde{\omega}$ and ω_0 are equidistants of the plane Λ^2.

Let ω_h be an equidistant of the plane Λ^2, and let ω_h lies in the interior domain with respect to the cycle $\widetilde{\omega}$. Then $\omega_h\,(S, r_h)$ corresponds under the inversion $\boldsymbol{I}$ to the hyperbolic cycle $\omega'_h\,(S, r'_h)$ of the plane $\widehat{H}$. In this case the module of the height of the equidistant $\omega_h(S, r_h)$ grows when the radius of the cycle ω'_h grows.

Let ω_h be an equidistant of the plane Λ^2, and let ω_h lies between the equidistants $\widetilde{\omega}$ and ω_0. Then $\omega_h\,(S, r_h)$ corresponds under the inversion $\boldsymbol{I}$ to the equidistant $\omega'_h\,(S, r'_h)$ which lies between the absolute of the plane $\widehat{H}$ and the equidistant ω_0. In this case the equidistants $\omega_h(S, r_h)$ and $\omega'_h(S, r'_h)$ together approach the equidistant ω_0 or move away from this equidistant.

6. The radius r'_h of the image ω'_h of a hyperbolic cycle ω_h of the plane H^2 under the inversion $\boldsymbol{I}$ can be expressed by the formulae:

$$\tanh\frac{r'_h}{\rho} = \mathrm{tg}^2\,\frac{r}{\rho}\coth\frac{r_h}{\rho}, \quad \tanh\frac{r'_h}{\rho} = \tanh\frac{\widetilde{r}}{\rho}\coth\frac{r_h}{\rho}, \quad \tanh\frac{r'_h}{\rho} = \tanh^2\frac{r_0}{\rho}\coth\frac{r_h}{\rho}.$$

Theorem 2. *The image l' of a line l under the inversion $\boldsymbol{I}$ with respect to the elliptic cycle ω of the plane $\widehat{H}$ is symmetric with respect to the common point of the line l and the base of ω. The curve l' contains the centre of the cycle ω, the absolute points of this cycle, the pole of the line l with respect to the cycle ω, and the points of intersection of the line l with ω.*

If the line l is the axis of an elliptic cycle ω of the plane $\widehat{H}$, then the image of l under the inversion $\boldsymbol{I}$ with respect to ω is couple of lines containing the line l and the base of the cycle ω.

If the line l passes through an absolute point X of the cycle ω, then the image of the line l under the inversion $\boldsymbol{I}$ is couple of lines containing the parabolic line connecting the center of cycle ω with its absolute point X, and the hyperbolic line connecting the second absolute point Y of ω with the point of the intersection of l and ω.

If the line l is the base of the cycle ω, then the image of l is the couple of parabolic lines with the common point at the cycle ω centre.

If the pole of the line l with respect to the cycle ω lies in the plane Λ^2, then the image of the line l is a hyperbola (see [5], [2, Section 2.3]*) of the plane $\widehat{H}$.*

If the pole of the line l with respect to the cycle ω lies on the absolute of the plane $\widehat{H}$, then the image of the line l is a hyperbola of the plane $\widehat{H}$ which has one branch and one interior domain.

If the pole of the line l with respect to the cycle ω lies in the plane $\widehat{H}$ and the point of the intersection of the line l with the base of ω lies in the plane Λ^2, then the image of the line l is a bihyperbola (see [5], [2, Section 2.3]*) of the plane $\widehat{H}$ which has one interior domain.*

If the pole of the line l with respect to the cycle ω and the point of the intersection of the line l with the base of ω lie in the plane $\widehat{H}$, then the image of the line l is a bihyperbola of the plane $\widehat{H}$ which has two interior domains.

References

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