Rational differential invariants for oriented primary visual cortex

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The classic experiments ([2],[3]) of David Hubel and Torsten Weisel published in 1959 (Gross Horwitz Prize, 1975, and Nobel Prize,1981) gave us understanding how neurons extract information about light cast on the retina. They investigated how neurons in the primary visual cortex respond when they moved a bright contour in retina. They noticed that neurons react only if the line passed a particular place of retina and with a certain orientation. Moreover, "sometimes a moving spot gave more activation for one direction than for the opposite",([3]).

Geometrically this result could be formulated in the following way. Assume that the retina is a 2-dimensional manifold M. Then for a given point of retina $a \in M$ "simple" neurons detect the point and an oriented line in the tangent plane, $p \subset T_aM$. Moreover, they constitute a "hypercolumn", which allows to detect any oriented line $p \subset T_aM$ at each point $a \in M$.

Oriented lines in tangent (or cotangent) planes to M (we'll work with cotangent planes) form so-called spherization of the cotangent bundle

$$\mathbf{S}\left(M\right) = \left(\mathbf{T}^{*}M \setminus 0\right)/\mathbb{R}^{+}$$

-the classical 3-dimensional contact manifold.

It is also known, that the simple neurons operate as filters (see [1], [4] for more details) on optic signal which could be considered as convolution with the Gaussian or Gabor filters. This means that, in addition to the contact structure, we have metric structure on M, (cf. [1]).

We consider the case of spherical geometry, $M = \mathbf{S}^2$ and $G = \mathbf{SO}\left(3, \mathbb{R}\right)$ - the special orthogonal group of rigid motions of the sphere.

Let $\mathbf{S}^2$ be riemann sphere equipped with stereographic coordinates (x, y) given by the stereographic projection and the Fubini -Study metric

$$g = \frac{4}{\left(1+x^2+y^2\right)^2}\left(dx^2+dy^2\right).$$

The Hamiltonian lift of Lie algebra $\mathfrak{so}_3\mathbb{R}$, which is the symmetry Lie algebra of metric g, to $S(M)$ is generated by the following vector fields:

$$\left(1+x^2-y^2\right)\partial_x + 2xy\partial_y + 2y\partial_u, \quad 2xy\partial_y + \left(1-x^2+y^2\right)\partial_y - 2x\partial_u, \quad x\partial_y - y\partial_x + \partial_u$$

in the standard canonical coordinates on $S(M)$.

The action of Lie group $\mathbf{SO}_3\left(\mathbb{R}\right)$ on $S(M)$ is free and transitive, and therefore we expect two independent 1-st order differential invariants for oriented distributions.

The standard Liouville 1-form gives us normed differential form on $S(M)$

$$\omega_1 = \frac{2}{1+x^2+y^2}\left(\cos u \; dx + \sin u \; dy\right)$$

and $\mathbf{SO}_3\left(\mathbb{R}\right)$-invariant orthonormal coframe $\langle\omega_1, \omega_2\rangle$ on $S(M)$ where

$$\omega_2 = \frac{2}{1+x^2+y^2}\left(-\sin u \; dx + \cos u \; dy\right).$$

The structure equations

$$\widehat{d}\omega_1 = J_1\omega_1 \wedge \omega_2, \quad \widehat{d}\omega_2 = J_2\omega_1 \wedge \omega_2$$

give us two differential $\mathfrak{so}_3(\mathbb{R})$-invariants of the 1-st order:

$$\begin{aligned} J_1 &= \frac{1+x^2+y^2}{2}\left(u_x \cos u + u_y \sin u\right) + \left(y \cos u - x \sin u\right), \\ J_2 &= \frac{1+x^2+y^2}{2}\left(-u_x \sin u + u_y \cos u\right) - \left(y \sin u + x \cos u\right). \end{aligned}$$

The frame dual to coframe $\langle \omega_1, \omega_2 \rangle$ provides us with two $\mathfrak{so}_3(\mathbb{R})$-invariant derivations

$$\nabla_1 = \frac{1+x^2+y^2}{2}\left(\cos u \,\frac{d}{dx} + \sin u \,\frac{d}{dy}\right), \quad \nabla_2 = \frac{1+x^2+y^2}{2}\left(-\sin u \,\frac{d}{dx} + \cos u \,\frac{d}{dy}\right),$$

with the following the syzygy relation for $\mathfrak{so}_3(\mathbb{R})$-invariants of oriented distributions has the form $J_{21} - J_{12} = J_1^2 + J_2^2 + 1$, where $J_{ij} = \nabla_i\left(J_j\right)$.

Taking invariant derivatives $\nabla_i\left(J_j\right)$ we get basic differential invariants J_{ij} of the second order in the form:

$$\begin{aligned} J_{11} &= \frac{1}{2}\left(A \cos 2u + B \sin 2u\right) + \frac{u_{xx}+u_{yy}}{2}\frac{\left(1+x^2+y^2\right)^2}{4}, \\ J_{12} &= \frac{1}{2}\left(B \cos 2u - A \sin 2u\right) - \frac{u_x^2+u_y^2}{2}\frac{\left(1+x^2+y^2\right)^2}{4} \\ &\quad + \frac{1+x^2+y^2}{2}\left(xu_y - yu_x - 1\right), \\ J_{22} &= -\frac{1}{2}\left(A \cos 2u + B \sin 2u\right) + \frac{u_{xx}+u_{yy}}{2}\frac{\left(1+x^2+y^2\right)^2}{4}, \end{aligned}$$

where

$$A = \frac{\left(1+x^2+y^2\right)^2}{4}\left(u_{xx} - u_{yy} + 2u_x u_y\right), \quad B = \frac{\left(1+x^2+y^2\right)^2}{4}\left(2u_{xy} - u_x^2 + u_y^2\right).$$

The following theorem describes the structure of differential $\mathfrak{so}_3(\mathbb{R})$-invariants of oriented distributions.

Theorem 1. (1) *Differential $\mathfrak{so}_3(\mathbb{R})$-invariants of oriented distributions on the unit sphere are generated by basic differential invariants J_1, J_2 and invariant derivations ∇_1, ∇_2, subjected to syzygy relation i.e. any $\mathfrak{so}_3(\mathbb{R})$-differential invariant is a rational function of invariants $\nabla^\mu\left(J_1\right)$ and $\nabla^\mu\left(J_2\right)$.*

(2) *In a neighborhood of regular point $\mathbf{SO}_3\left(\mathbb{R}\right)$-orbit of oriented distribution on the unit sphere is uniquely defined by invariants J_{11}, J_{12}, J_{22} as functions in J_1, J_2 .*

References

[1] Alekseevsky, Dmitri, Light, Eye, Brain. Geometric Problems in vision, Proceedings of GTS, ed. B. Bidabad, 8th Seminar on Geometry and Topology, p.10-32 Tehran, 2017

[2] Hubel D., Eye, Brain and Vision, "Scientific American" Library, 1995

[3] Hubel D.H. and Wiesel T.N. Receptive fields of single neurons in the cat's striate cortex, J. Physiol. (I959) I48, 574-59I

[4] Petitot J.,Neurogeometry of neural functional architectures, Chaos, Solitons & Fractals, 50 (2013), 75–92

[5] N. Konovenko, V. Lychagin Invariants for primary visual cortex. *Journal of differential geometry and its applicat*, (in print)