An extension of Möbius–Lie geometry with conformal ensembles of cycles

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Lie sphere geometry in the simplest planar setup unifies circles, lines and points—all together called *cycles* in this setup [8]. Symmetries of Lie spheres geometry include (but are not limited to) fractional linear transformations (FLT) of the form:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : \ x \mapsto \frac{ax+b}{cx+d}, \qquad \text{where } \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} \neq 0. \tag{1}$$

There is a natural set of FLT-invariant geometric relations between cycles (to be orthogonal, to be tangent, etc.) and the restriction of Lie sphere geometry to invariants of FLT is called *Möbius–Lie geometry*. Thus, an ensemble of cycles, structured by a set of such relations, will be mapped by FLT to another ensemble with the same structure.

It is convenient to deals with cycles through Fillmore–Springer–Cnops construction (FSCc) which also has a long history, see [8, § 4.2]. Compared to a plain analytical treatment [3, Ch. 3], FSCc is much more efficient and conceptually coherent in dealing with FLT-invariant properties of cycles. For example, the inner product $\langle C_1, C_2 \rangle$ of cycles C_1 and C_2 is equal to zero if and only if cycles are orthogonal, Other properties, e.g. tangency, has a natural presentation as well, see [8, Ch. 4–5] and [9].

It was shown recently that ensembles of cycles with certain FLT-invariant relations provide helpful parametrisations of new objects as follows.

Example 1. (1) The Poincaré extension of Möbius transformations from the real line to the upper half-plane of complex numbers is described by a triple of cycles $\{C_1, C_2, C_3\}$ such that:
   (a) C_1 and C_2 are orthogonal to the real line;
   (b) $\langle C_1, C_2 \rangle^2 \leq \langle C_1, C_1 \rangle \langle C_2, C_2 \rangle$;
   (c) C_3 is orthogonal to any cycle in the triple including itself.

   A modification [10] with ensembles of four cycles describes an extension from the real line to the upper half-plane of complex, dual or double numbers. The construction can be generalised to arbitrary dimensions.

(2) A parametrisation of loxodromes is provided by a triple of cycles $\{C_1, C_2, C_3\}$ such that [12]:
   (a) C_1 is orthogonal to C_2 and C_3;
   (b) $\langle C_2, C_3 \rangle^2 \geq \langle C_2, C_2 \rangle \langle C_3, C_3 \rangle$.

   Then, main invariant properties of Möbius–Lie geometry, e.g. tangency of loxodromes, can be expressed in terms of this parametrisation [12].

(3) A continued fraction is described by an infinite ensemble of cycles (C_k) such that [2]:
   (a) All C_k are touching the real line (i.e. are *horocycles*);
   (b) (C_1) is a horizontal line passing through $(0, 1)$;
   (c) C_{k+1} is tangent to C_k for all $k > 1$.

   This setup was extended in [10] to several similar ensembles. The key analytic properties of continued fractions—their convergence—can be linked to asymptotic behaviour of such an infinite ensemble [2].

(4) An important example of an infinite ensemble is provided by the representation of an arbitrary wave as the envelope of a continuous family of spherical waves. A finite subset of spheres can be used as an approximation to the infinite family. Then, discrete snapshots of time evolution

of sphere wave packets represent a FLT-covariant ensemble of cycles. This and further physical applications of FLT-invariant ensembles may be looked at [5].

Definition 2. The extend Möbius–Lie geometry considers ensembles of cycles interconnected through FLT-invariant relations.

Naturally, "old" objects—cycles—are represented by simplest one-element ensembles without any relation. The paper [9] provides conceptual foundations of such extension and demonstrates its practical implementation as a CPP library **figure**[1]. Interestingly, the development of this library shaped the general approach, which leads to specific realisations in [11, 10, 12].

More specifically, the library **figure** manipulates ensembles of cycles (quadrics) interrelated by certain FLT-invariant geometric conditions. The code is build on top of the previous library **cycle** [7, 8, 6], which manipulates individual cycles within the GiNaC [1] computer algebra system. Thinking an ensemble as a graph, one can say that the library **cycle** deals with individual vertices (cycles), while **figure** considers edges (relations between pairs of cycles) and the whole graph. Intuitively, an interaction with the library **figure** reminds compass-and-straightedge constructions, where new lines or circles are added to a drawing one-by-one through relations to already presented objects (the line through two points, the intersection point or the circle with given centre and a point).

It is important that both libraries are capable to work in spaces of any dimensionality and metrics with an arbitrary signatures: Euclidean, Minkowski and even degenerate. Parameters of objects can be symbolic or numeric, the latter admit calculations with exact or approximate arithmetic. Drawing routines work with any (elliptic, parabolic or hyperbolic) metric in two dimensions and the euclidean metric in three dimensions.

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[1] All described software is licensed under GNU GPLv3 [4].