Class groups of rings with divisor theory, L-functions and moduli spaces

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The study of class groups of rings and corresponding schemes ts an actual scientific problem (see [1, 2] and references therein). For regular local rings, according to the Auslander-Buchsbaum theorem, the (divisors) class group is trivial. But in most interesting cases the group is nontrivial. The Heegner approach, together with the results of Weber, Birch, Baker and Stark, makes it possible to calculate and even parametrize rings with a given (small) class number in some cases. Let R be a commutative ring with identity for which there exists the theory of divisors [2]. The order of the class group is calculated on the basis of the use of L-functions. We investigate one of the aspects of this problem, consisting in finding the moduli spaces of elliptic curves defined over the rings R with the given class number.

Problem 1. To investigate the case of elliptic curves over rings of integers of quadratic fields (rings of integers $\mathcal{O}$ of quadratic algebraic extensions k of the field of rational numbers $\mathbb{Q}$) with a small class number, see [2].

We present an elementary introduction to this problem and give the moduli spaces as trivial bundles over affine part of the groups of rational points of some elliptic curves over the ring of integers $\mathbb{Z}$. Below we present parameter spaces and moduli for class number one and two. Let

$$E : y^2 = x^3 + ax + b,\ Disc(E) = 4a^3 + 27b^2, Disc(E) \neq 0, \quad (*)$$

be an elliptic curve over the ring $\mathcal{O}$. Let A_1 be the affine part of the group of rational points over $\mathbb{Z}$ of the Heegner elliptic curve $y^2 = 2x(x^3 + 1)$. With results by Heegner, Deuring, Birch, Baker, Stark, Kenku, Abrashkin, we deduce

Proposition 2. *Let $\mathcal{O}$ be the ring of integers of the imaginary quadratic field with class number one. Then the parameter space of elliptic curves of the form (*) is the trivial bundle*

$$(\mathcal{O} \times \mathcal{O}/(Disc(E) = 0)) \times A_1.$$

Proposition 3. *Let k be the imaginary quadratic field with class number one. Then the moduli space of elliptic curves of the form (*) is the trivial bundle*

$$k \times A_1.$$

Let A_2 be the affine part of the group of rational points over $\mathbb{Z}$ of the elliptic curve $X^3+3X = -Y^2$, let A_3 be the affine part of the group for the elliptic curve $X^3 - 3X = 2Y^2$, and A_4 respectively for $9X^4 - 1 = 2Y^2$.

Proposition 4. *Let $\mathcal{O}$ be the ring of integers of the imaginary quadratic field with class number two. Then the parameter spaces of elliptic curves of the form (*), without an exceptional case, are trivial bundles*

$$(\mathcal{O} \times \mathcal{O}/(Disc(E) = 0)) \times A_2,\ (\mathcal{O} \times \mathcal{O}/(Disc(E) = 0)) \times A_3,\ (\mathcal{O} \times \mathcal{O}/(Disc(E) = 0)) \times A_4.$$

Proposition 5. *Let k be the imaginary quadratic field with class number two. Then the moduli spaces of elliptic curves of the form (*), without an exceptional case, are the trivial bundles*

$$k \times A_2,\ k \times A_3,\ k \times A_4.$$

In the article [3] the main object of the investigation is the $L-$function of the family of superelliptic curves over $K = \mathbb{F}_q(t)$ and their models $\mathcal{E} \to \mathbb{P}^1$. In the article [4] authors investigate some aspects of the problem in the framework of the theory of Heegner-Stark (Darmon) and Darmon-like points in elliptic curves E over $\mathbb{Q}$ of conductor N with a prime p such that $N = pDM$, where D is the product of even (possible zero) distinct primes and $(D, M) = 1$, and develop the (co)homological techniques for effective construction of the quaternionic Darmon points on $E(K_p)$, where K_p is the $p-$adic upper half plane. If there is enough time, we plan to discuss the possible extension of the research in these directions [3, 4].

References

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