Dual modules over Steenrod algebra 2

Alexander N. Vasilchenko
(19 Cheluskintsev, a.32, Samara)
E-mail: vass-alexandr@yandex.ru

Article [3] studies structures of modules over Steenrod algebra A and their duals in dual Steenrod algebra A^* [2], [4]. [1] studies modules $A(n)$ over A generated by annihilators of cohomology classes with degrees no greater then n. $A(n)^+$ is defined as annihilators of cohomology operations with excess greater then n [6]. In A^*, $A(n)^*$ is the corresponding dual module of $A(n)$. $A(n)^+$ as vector space over Z_p is generated by all monomials of A^* with multiplicity no greater then n [5], [6].

This work studies structure of modules $B(n) = (A(n-1)/A(n))^*$. $B(n)$ as a Z_p vector space has a basis formed by all monomials with multiplicity n in dual Steenrod algebra A^* [6], and $B(n)$ can be considered both left and right over A^*.

The result is stated here:

Theorem 1 (**Properties of quotient A^*-modules** $(A(n-1)/A(n))^*$)**.** 1) *$B(n)$ is a graded Hopf comodule over Steenrod algebra A^* with comultiplication*

$$\phi_n^* : B(n) \to A^* \bigotimes B(n), \qquad \phi_n^*([\alpha]) = \sum_i \alpha_i' \bigotimes [\alpha_i'']$$

induced by comultiplication in comodule $A(n)^+$, with homomorphism property

$$\begin{aligned}\phi_n^*([\alpha] * [\beta]) &= \phi_n^*([\alpha\beta]) \\ &= (\psi^* \otimes \psi_n^*)(Id_{A^*} \otimes T \otimes Id_{B(n_2)})(\phi_{n_1}^*([\alpha]) \otimes \phi_{n_2}^*([\beta])) \\ &\overset{def}{=} \phi_{n_1}^*([\alpha]) * \phi_{n_2}^*([\beta])\end{aligned}$$

where $\psi_{n_1+n_2}^* : B(n_1) \bigotimes B(n_2) \to B(n_1+n_2)$ is a multiplication defined by

$$\psi_{n_1+n_2}^*([\alpha] \otimes [\beta]) = [\psi^*(\alpha \otimes \beta)] = [\alpha\beta] = [\alpha] * [\beta]$$

induced by multiplication ψ^* in A^*, T is a transposition, $[\alpha]$ in $B(n_1)$, and $[\beta]$ in $B(n_2)$.

2) *$B(n) = \bigoplus_s B(n)^s$ is the direct sum of Hopf comodules*

$$B(n)^s = Span\{\tau_0^{s_0}\tau_1^{s_1}\tau_2^{s_2}\dots\xi_1^{r_1}\xi_2^{r_2}\xi_3^{r_3}\dots \in A^* | \sum_i s_i = s\}$$

3) *$B(n)_t = \bigoplus_s B(n)_t^s$ is the direct sum of comodules $B(n)_t = (A(n)^+ \cap A_t^*)/(A(n-1)^+ \cap A_t^*)$ defined on the filtration of dual Steenrod algebra A^* by Hopf subalgebras*

$$A_{-1}^* \subset A_0^* \subset A_1^* \subset \dots \subset A_n^* \subset A_{n+1}^* \subset \dots A^*,$$

where $A_t^* = Z_p\{\xi_1, \xi_2, \dots\} \bigotimes E\{\tau_0, \tau_1 \dots \tau_t\}$ and $A_{-1}^* = Z_p\{\xi_1, \xi_2 \dots\}$. The restrictions of the comultiplication and multiplication (1) on the filtration are well defined maps:

$$\phi_{n,t}^* : B(n)_t \to A^* \otimes B(n)_t, \qquad \text{and} \qquad \psi_{n_1 n_2,t}^* : B(n_1)_t \otimes B(n_2)_t \to B(n_1+n_2)_t.$$

4) *Any $[\alpha]$ in $B(n)_t$ has unique form $[\alpha] = [\beta] + [\gamma] * [\tau_t]$ where $[\beta] \in B(n)_{t-1}$ and $[\gamma] \in B(n-1)_{t-1}$, and there are homomorphisms of comodules $i_{n,t}$ and $\pi_{n,t}$ such that $i_{n,t}([\beta]) = [\beta]$, and $\pi_{n,t}([\alpha]) = [\gamma]$,*

and the following diagram with exact rows commutes

$$\begin{array}{ccccccccc}
0 & \longrightarrow & B(n)_{t-1} & \xrightarrow{i_{n,t}} & B(n)_t & \xrightarrow{\pi_{n,t}} & B(n-1)_{t-1} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \phi^*_{n,t-1}} & & \downarrow{\scriptstyle \phi^*_{n,t}} & & \downarrow{\scriptstyle \phi^*_{n-1,t-1}} & & \\
0 & \longrightarrow & A^* \otimes B(n)_{t-1} & \xrightarrow{Id \otimes i_{n,t}} & A^* \otimes B(n)_t & \xrightarrow{Id \otimes \pi_{n,t}} & A^* \otimes B(n-1)_{t-1} & \longrightarrow & 0
\end{array}$$

Rerefences

[1] H. Cartan. Algebres d'Eilenberg-MacLane at Homotopie. *Seminare Cartan ENS*, 7e, 1954–1955.

[2] J. Milnor. The Steenrod algebra and its dual. *Annals of Mathematics*, 67 : 150–171, 1958.

[3] J. Milnor, J. Moore. On the structure of Hopf algebras. *Annals of Mathematics*, 81 : 211–264, 1965.

[4] N. Steenrod, D. B. A. Epstein. *Cohomological Operations*, Princeton University Press, 1962.

[5] А. Н. Васильченко. Свойства дуальных модулей над алгеброй Стинрода. Abstracts of the International Conference "Gemetry in Odessa" – 2014, Odessa the 26th of May – the 31st of May 2014: p.26

[6] А. Н. Васильченко. Свойство дуальных модулей над алгеброй Стинрода. *Вестник СамГУ* 7(118) : 9–16, 2014.