Integrable systems with dissipation on the tangent bundle of two-dimensional manifold

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We study nonconservative systems for which the usual methods of the study, e.g., Hamiltonian systems, are inapplicable. Thus, for such systems, we must "directly" integrate the main equation of dynamics. We generalize previously known cases and obtain new cases of the complete integrability in transcendental functions of the equation of dynamics of a four-dimensional rigid body in a nonconservative force field.

We obtain a series of complete integrable nonconservative dynamical systems with nontrivial symmetries. Moreover, in almost all cases, all first integrals are expressed through finite combinations of elementary functions; these first integrals are transcendental functions of their variables. In this case, the transcendence is understood in the sense of complex analysis, when the analytic continuation of a function into the complex plane has essentially singular points. This fact is caused by the existence of attracting and repelling limit sets in the system (for example, attracting and repelling focuses).

We detect new integrable cases of the motion of a rigid body, including the classical problem of the motion of a multi-dimensional spherical pendulum in a flowing medium.

This activity is devoted to general aspects of the integrability of dynamical systems with variable dissipation. First, we propose a descriptive characteristic of such systems. The term "variable dissipation" refers to the possibility of alternation of its sign rather than to the value of the dissipation coefficient (therefore, it is more reasonable to use the term "sign-alternating") [1, 2].

We introduce a class of autonomous dynamical systems with one periodic phase coordinate possessing certain symmetries that are typical for pendulum-type systems. We show that this class of systems can be naturally embedded in the class of systems with variable dissipation with zero mean, i.e., on the average for the period with respect to the periodic coordinate, the dissipation in the system is equal to zero, although in various domains of the phase space, either energy pumping or dissipation can occur, but they balance to each other in a certain sense. We present some examples of pendulum-type systems on lower-dimension manifolds from dynamics of a rigid body in a nonconservative field.

Then we study certain general conditions of the integrability in elementary functions for systems on the two-dimensional plane and the tangent bundles of a one-dimensional sphere (i.e., the two-dimensional cylinder) and a two-dimensional sphere (a four-dimensional manifold). Therefore, we propose an interesting example of a three-dimensional phase portrait of a pendulum-like system which describes the motion of a spherical pendulum in a flowing medium (see also [3, 4]).

To understand the difficulty of problem resolved, for instance, let us consider the spherical pendulum (ψ and θ — the coordinates of point on the sphere where the pendulum is defined) in a jet flow. Then the equations of its motion are

$$\ddot{\theta} + (b_* - H_1^*)\dot{\theta}\cos\theta + \sin\theta\cos\theta - \dot{\psi}^2\frac{\sin\theta}{\cos\theta} = 0, \tag{1}$$

$$\ddot{\psi} + (b_* - H_1^*)\dot{\psi}\cos\theta + \dot{\theta}\dot{\psi}\frac{1+\cos^2\theta}{\cos\theta\sin\theta} = 0,\ b_* > 0,\ H_1^* > 0, \tag{2}$$

and the phase pattern of the eqs. (1), (2) is on the Fig. 0.1.

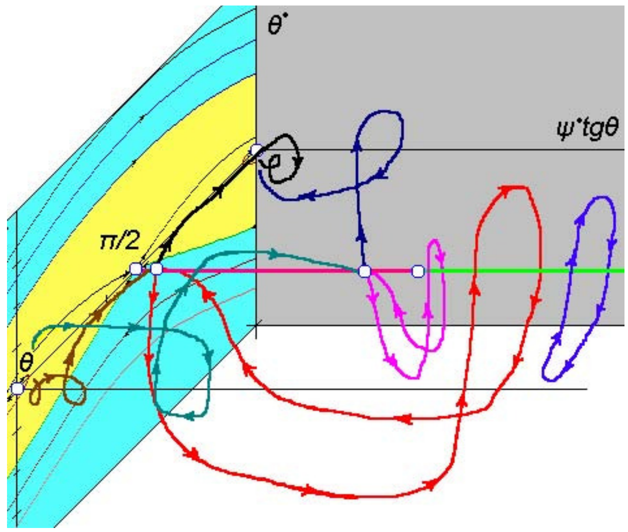

FIGURE 0.1. Phase pattern of spherical pendulum in a jet flow.

The assertions obtained in the work for variable dissipation system are a continuation of the Poincare–Bendixon theory for systems on closed two-dimensional manifolds and the topological classification of such systems.

The problems considered in the work stimulate the development of qualitative tools of studying, and, therefore, in a natural way, there arises a qualitative variable dissipation system theory.

Following Poincare, we improve some qualitative methods for finding key trajectories, i.e., the trajectories such that the global qualitative location of all other trajectories depends on the location and the topological type of these trajectories. Therefore, we can naturally pass to a complete qualitative study of the dynamical system considered in the whole phase space. We also obtain condition for existence of the bifurcation birth stable and unstable limit cycles for the systems describing the body motion in a resisting medium under the streamline flow around. We find methods for finding any closed trajectories in the phase spaces of such systems and also present criteria for the absence of any such trajectories. We extend the Poincare topographical plane system theory and the comparison system theory to the spatial case. We study some elements of the theory of monotone vector fields on orientable surfaces.

Rerefences

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