Some extremal and structural properties of finite ultrametric spaces

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We describe some extremal and structural properties of finite ultrametric spaces which have the strictly n-ary representing trees and the representing trees with injective internal labeling.

Recall some definitions. The *spectrum* of an ultrametric space (X, d) is the set

$$\operatorname{Sp}(X) = \{d(x, y) \colon x, y \in X\}.$$

For a graph $G = (V, E)$, the sets $V = V(G)$ and $E = E(G)$ are called *the set of vertices (or nodes)* and *the set of edges*, respectively. Let (X, d) be an ultrametric space with $|X| \geqslant 2$ and the spectrum $\operatorname{Sp}(X)$ and let $r \in \operatorname{Sp}(X)$ be nonzero. Define by $G_{r,X}$ a graph for which $V(G_{r,X}) = X$ and

$$(\{u, v\} \in E(G_{r,X})) \Leftrightarrow (d(u, v) = r).$$

Denote by G' the subgraph of the graph G obtained from G by deleting of isolated vertices. With every finite ultrametric space (X, d) we can associate a labeled rooted tree T_X, see [1]. By $\mathbf{B}_X$ we denote the set of all balls of the space X.

Theorem 1. *Let (X, d) be a finite nonempty ultrametric space. The following conditions are equivalent.*

(i) The diameters of different nonsingular balls are different.
(ii) The internal labeling of T_X is injective.
(iii) $G'_{r,X}$ is a complete multipartite graph for every $r \in \operatorname{Sp}(X) \setminus \{0\}$.
(iv) The equality $|\mathbf{B}_X| = |X| + |\operatorname{Sp}(X)| - 1$ holds.

Let (X, d) be an ultrametric space. Recall that balls $B_1, \ldots, B_k$ in (X, d) are equidistant if there is $r > 0$ such that $d(x_i, x_j) = r$ holds whenever $x_i \in B_i$ and $x_j \in B_j$ and $1 \leqslant i < j \leqslant k$.

Theorem 2. *Let (X, d) be a finite nonempty ultrametric space and let $n \geqslant 2$ be integer. The following conditions are equivalent.*

(i) T_X is strictly n-ary.
(ii) For every nonzero $r \in \operatorname{Sp}(X)$ the graph $G'_{r,X}$ is the union of p complete n-partite graphs, where p is a number of all internal nodes of T_X labeled by r.
(iii) For every nonsingular ball $B \in \mathbf{B}_X$ there are the equidistant balls $B_1, ..., B_n \in \mathbf{B}_X$ such that $B = \bigcup\limits_{j=1}^{n} B_j$ and $\operatorname{diam} B_j < \operatorname{diam} B$ for every $j \in \{1, \ldots, n\}$.
(iv) The equality $(n - 1)|\mathbf{B}_Y| + 1 = n|Y|$ holds for every ball $Y \in \mathbf{B}_X$.

Let T be a rooted tree and let v be a node of T. Denote by $\delta^+(v)$ the out-degree of v, i.e., $\delta^+(v)$ is the number of children of v, and write

$$\Delta^+(T) = \max_{v \in V(T)} \delta^+(v),$$

i.e., $\Delta^+(T)$ is the maximum out-degree of $V(T)$. It is clear that $v \in V(T)$ is a leaf of T if and only if $\delta^+(v) = 0$. Moreover, T is strictly n-ary if and only if the equality

$$\delta^+(v) = n$$

holds for every internal node v of T. Let us denote by $I(T)$ the set of all internal nodes of T.

Lemma 3. *The inequality*

$$|V(T)| \leqslant \Delta^+(T)|I(T)| + 1$$

holds for every rooted tree T. If $|V(T)| \geqslant 2$, then this inequality becomes the equality if and only if T is strictly n-ary with $n = \Delta^+(T)$.

Corollary 4. *The inequality*

$$|\mathbf{B}_X| \geqslant \frac{\Delta^+(T_X)|X| - 1}{\Delta^+(T_X) - 1}$$

holds for every finite nonempty ultrametric space (X, d). This inequality becomes an equality if and only if T_X is a strictly n-ary rooted tree with $n = \Delta^+(T_X)$.

Proposition 5. *Let (X, d) be a finite ultrametric space with $|X| \geq 2$. Then the inequality*

$$2|\mathbf{B}_X| \geqslant |\operatorname{Sp}(X)| + \frac{2\Delta^+(T_X)|X| - \Delta^+(T_X) - |X|}{\Delta^+(T_X) - 1}$$

holds. This inequality becomes an equality if and only if T_X is a strictly n-ary rooted tree with $n = \Delta^+(T_X)$ and with the injective internal labeling.

Corollary 6. *Let (X, d) be a finite ultrametric space with $|X| \geqslant 2$ and let $n = \Delta^+(T_X)$. The following conditions are equivalent.*

(i) *T_X is a strictly n-ary tree with injective internal labeling.*
(ii) *$G'_{r,X}$ is complete n-partite graph for every $r \in \operatorname{Sp}(X)$.*
(iii) *The equality*

$$2|\mathbf{B}_X| = |\operatorname{Sp}(X)| + \frac{2\Delta^+(T_X)|X| - \Delta^+(T_X) - |X|}{\Delta^+(T_X) - 1}$$

holds.

Rerefences

[1] Evgenii A. Petrov, Aleksei A. Dovgoshey. On the Gomory–Hu Inequality. *Journal of Mathematical Sciences*, 198(4): 392–411, 2014.

[2] O. Dovgoshey, E. Petrov, H.-M. Teichert. Extremal properties of finite ultrametric spaces and invariants of their isometries. arXiv:1610.08282v1, 2016.