Landau-type inequalities for curves on Riemannian manifolds

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Let for a natural number n a function $f(\cdot) \in \mathrm{C}^n(\mathbb{R} \mapsto \mathbb{R})$ satisfy the inequalities

$$\|f(\cdot)\|_\infty := \sup_{t\in\mathbb{R}} |f(t)| < \infty, \quad \left\|f^{(n)}(\cdot)\right\|_\infty < \infty.$$

In the case where $n = 2$, the famous Landau –Hadamard inequality reads

$$\|f'(\cdot)\|_\infty \le \sqrt{2\,\|f(\cdot)\|_\infty\,\|f''(\cdot)\|_\infty}.$$

In the general case $n \ge 2$, $1 \le k < n$, Kolmogorov determined the best constants $C_{n,k}$ for inequality

$$\left\|f^{(k)}(\cdot)\right\|_\infty \le C_{n,k}\,\|f(\cdot)\|^{1-k/n}\left\|f^{(n)}(\cdot)\right\|_\infty^{k/n}.$$

(see, e.g., [1]). The goal of the present report is to discuss how the above inequalities can be generalized for the case of mappings taking values in Riemannian manifolds.

Let $(\mathcal{M}, \mathfrak{g} = \langle\cdot,\cdot\rangle)$ be a smooth complete Riemannian manifold with the metric tensor $\mathfrak{g}$, and let ∇ be the Levi-Civita connection with respect to $\mathfrak{g}$. For a given smooth mapping $x(\cdot) : I \mapsto \mathcal{M}$ of an interval $I \subset \mathbb{R}$ and for a smooth vector field $\xi(\cdot) : I \mapsto T\mathcal{M}$ along $x(\cdot)$, denote by $\nabla_{\dot x}\xi(t)$ the covariant derivative of $\xi(\cdot)$ along the tangent vector $\dot x(t) \in T_{x(t)}\mathcal{M}$ at the point $t \in I$, and by $\nabla_{\dot x}^k$ the k-th iterate of $\nabla_{\dot x}$. (Here $T\mathcal{M} = \bigsqcup_{x\in\mathcal{M}} T_x\mathcal{M}$ stands for the total space of the tangent bundle with natural projection $\pi(\cdot) : T\mathcal{M} \mapsto \mathcal{M}$, and $T_x\mathcal{M} = \pi^{-1}(x)$ denotes the tangent space to $\mathcal{M}$ at x.)

For a smooth function $U(\cdot) : \mathcal{M} \mapsto \mathbb{R}$ denote by $\nabla U(x) \in T_x\mathcal{M}$ and by $H_U(x) : T_x\mathcal{M} \mapsto T_x\mathcal{M}$, respectively, the gradient vector and the Hesse form of $U(\cdot)$ at point x (by the definition

$$\langle H_U(x)\xi, \eta\rangle = \langle \nabla_\xi \nabla U(x), \eta\rangle$$

for any $x \in \mathcal{M}$ and any $\xi, \eta \in T_x\mathcal{M}$). Define the natural norm for tangent vector ξ as $\|\xi\| := \sqrt{\langle\xi,\xi\rangle}$ and the norm for vector field $\xi(\cdot)$ along $x(\cdot)$ as

$$\|\xi(\cdot)\|_\infty := \sup_{t\in\mathbb{R}} \|\xi(t)\|.$$

We obtain the following Landau-type inequality.

Theorem 1. *Let $x(\cdot) : \mathbb{R} \mapsto \mathcal{M}$ be a smooth mapping such that*

$$\|\nabla_{\dot x}\dot x(\cdot)\|_\infty < \infty.$$

Suppose that there exists a smooth function $U(\cdot) : \mathcal{M} \mapsto \mathbb{R}$ satisfying the inequalities

$$\sup_{t\in\mathbb{R}} U \circ x(t) < \infty, \quad \|\nabla U \circ x(\cdot)\|_\infty < \infty,$$

and

$$\lambda := \inf_{t\in\mathbb{R}} \min\left\{\langle [H_U \circ x(t)]\,\xi, \xi\rangle : \xi \in T_{x(t)}\mathcal{M},\ \|\xi\| = 1\right\} > 0.$$

Then

$$\|\dot x(\cdot)\|_\infty \le C\sqrt{\|\nabla U \circ x(\cdot)\|_\infty\,\|\nabla_{\dot x}\dot x(\cdot)\|_\infty/\lambda}$$

where the constant C does not exceed the positive root of the equation $\zeta^3 - 3\zeta = 1$. In particular, $C < 1.87939$.

Remark 2. If $\mathcal{M} = \mathbb{R}^d$ and $U(x) := \|x\|^2/2$, then $\lambda = 1$, and Theorem 1 leads to the Landau inequality with the constant C somewhat greater then the best one $C_{2,1} = \sqrt{2}$.

If the closure of $x(\cdot)$ is a compact subset $\mathcal{K}$ of a domain $\mathcal{D} \subset \mathcal{M}$ and there exists a point $x_0 \in \mathcal{D} \setminus \mathcal{K}$ such that the distance function

$$\rho(\cdot, x_0) : \mathcal{D} \setminus \{x_0\} \mapsto (0, \infty)$$

is smooth, then Theorem 1 holds true for $U(x) := \rho^2(x, x_0)$.

It turns out that it is much easier to obtain a counterpart of the Landau – Kolmogorov inequality for vector fields along mappings.

Theorem 3. *Let $\xi(\cdot) : \mathbb{R} \mapsto T\mathcal{M}$ be a smooth vector field along a smooth mapping $x(\cdot) : \mathbb{R} \mapsto \mathcal{M}$ and let $n \geq 2$ be a natural number. Suppose that*

$$\|\xi(\cdot)\|_\infty < \infty, \quad \|\nabla^n_{\dot{x}}\xi(\cdot)\|_\infty < \infty.$$

Then for any natural $k < n$ there holds the inequality

$$\left\|\nabla^k_{\dot{x}}\xi(\cdot)\right\|_\infty \leq C_{n,k} \|\xi(\cdot)\|_\infty^{1-k/n} \|\nabla^n_{\dot{x}}\xi(\cdot)\|_\infty^{k/n}$$

where $C_{n,k}$ are the Kolmogorov constants.

Rerefences

[1] Steven R. Finch. *Mathematical constants*, volume 94 of *Encyclopedia of Mathematics and its Applications*. Cambridge: Cambridge University Press, 2003.