Archimedean copula functions and their some algebraic properties with applications

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The word "copula" was first used in a mathematical sense by Sklar (1959, see [3]). But the functions themselves predate the use of the term, appearing in the work of Hoeffding, Fréchet, Dall'Aglio, and many others(see, [2]). Over the past forty years or so, copulas have played an important role in finance, engineering, bio-medical research, hydrology, climate and weather research, social research, econometrics, insurance, statistical research, dynamical systems and many areas. Axiomatically, a copula can be defined as follows.

Definition 1. A two-dimensional copula C is a mapping from $I^2 = [0,1] \times [0,1]$ to $I = [0,1]$ which satisfies the following three conditions:

1. $C(u,0) = C(0,u) = 0$ for every $u \in [0,1]$;
2. $C(u,1) = C(1,u) = u$ for every $u \in [0,1]$;
3. $C(u_2,v_2) - C(u_1,v_2) - C(u_2,v_1) + C(u_1,v_1) \geq 0$ for every $u_1, v_1, u_2, v_2 \in [0,1]$ satisfying $u_1 \leq u_2$, $v_1 \leq v_2$.

For example of copulas are the product or independence copula $C_\perp(u,v) = uv$, minimum copula $C_{min}(u,v) = min(u,v)$ and maximal copula $C_{max}(u,v) = max(u+v-1,0)$. In figure 1 we present the graphs of the copulas $C_\perp$, C_{min} and C_{max}.

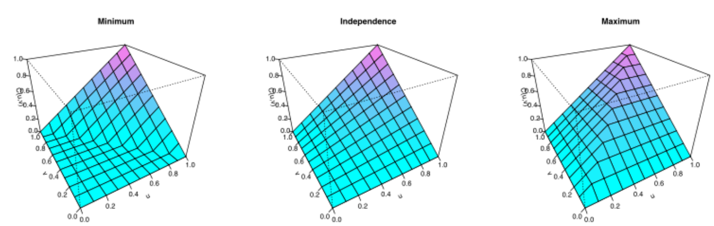

FIGURE 1.1. Graphs of the copulas $C_\perp$, C_{min} and C_{max}.

In this work, we discuss an important class of copulas known as Archimedean copulas. These copulas find a wide range of applications for a number of reasons: 1) the ease with which they can be constructed; 2) the great variety of families of copulas which belong to this class; and 3) the many nice properties possessed by the members of this class.

Let φ be a continuous, strictly decreasing function from $[0,1]$ to $[0,\infty]$such that $\varphi(1) = 0$.

Definition 2. The pseudo-inverse of φ is the function $\varphi^{[-1]}$ with $Dom\varphi^{[-1]} = [0,\infty]$ and $Ran(f)$ given by

$$\varphi^{[-1]}(t) = \begin{cases} \varphi^{-1}(t), & 0 < t < \varphi(0), \\ 0, & \varphi(0) \leq t \leq \infty. \end{cases} \tag{1}$$

Note that $\varphi^{[-1]}$ is continuous and no increasing on $[0,\infty]$, and strictly decreasing on $[0,\varphi(0)]$. Furthermore,

$$\varphi\big(\varphi^{[-1]}(t)\big)=\begin{cases}t, & 0<t<\varphi(0),\\ 0, & \varphi(0)\le t\le\infty\end{cases}=\min\left(t,\varphi\left(0\right)\right)$$

If $\varphi\left(0\right)=\infty$, then $\varphi^{[-1]}=\varphi^{-1}$.

Definition 3. Copulas of the form

$$C\left(u,v\right)=\varphi^{[-1]}\left(\varphi\left(u\right)+\varphi\left(v\right)\right), \tag{2}$$

are called Archimedean copulas, where the function φ is called a generator of the copula $\varphi\left(1\right)=0$.

Theorem 4. *Let φ be a continuous, strictly decreasing function from I to $[0,\infty]$ such that $\varphi(1)=0$, and let $\varphi^{[-1]}$ be the pseudo-inverse of φ defined by (2)(taken from* [2]*). Then the function C given by (3) is a copula if and only if φ is convex.*

We conclude this work with two theorems concerning some algebraic properties of Archimedean copulas.

Theorem 5. *Let C be an Archimedean copula (taken from* [2]*) with generator φ. Then:*

1. *C is symmetric, i.e., $C(u,v)=C(v,u)$ for all u,v in I;*
2. *C is associative, i.e., $C(C(u,v),w)=C(u,C(v,w))$ for all u,v,w in I;*
3. *If $c>0$ is any constant, then $c\varphi$ is also a generator of C.*

For convenience, let Ω denote the set of continuous strictly decreasing convex functions φ from I to $[0,\infty]$ with $\varphi(1)=0$. By now the reader is surely wondering about the meaning of the term "Archimedean" for these copulas. Recall the Archimedean axiom for the positive real numbers: If a,b are positive real numbers, then there exists an integer n such that $na>b$. An Archimedean copula behaves like a binary operation on the interval I, in that the copula C assigns to each pair u,v in I a number $C(u,v)$ in I. From Theorem 5, we see that the "operation" C is commutative and associative, and preserves order, i.e., $u_1\le u_2$ and $v_1\le v_2$ implies $C(u_1,v_1)\le C(u_2,v_2)$. Algebraists call (I,C) an ordered Abelian semigroup. For any u in I, we can define the C-powers u_C^n of u recursively: $u_C^1=u$, and $u_C^{n+1}=C(u,u_C^n)$, note that u_C^2 belongs to the diagonal section $\delta_C(u)$ of C. The version of the Archimedean axiom for (I,C) is, for any two numbers u,v in $(0,1)$, there exists a positive integer n such that $u_C^n<\nu$. The next theorem shows that Archimedean copulas satisfy this version of the Archimedean axiom and hence merit their name. The term "Archimedean" for these copulas was introduced in Ling (1965, see [2]).

Theorem 6. *Let C be an Archimedean copula (taken from* [2]*) generated by φ in Ω. Then for any u,v in I, there exists a positive integer n such that $u_C^n<\nu$.*

In many applications, the random variables(r.v.-s) of interest represent the lifetimes of individuals or objects in some population. In survival analysis our interest focuses on a nonnegative r.v.-s denoting death times of biological organisms or failure times of mechanical systems. A difficulty in the analysis of survival data is the possibility that the survival times can be subjected to random censoring by other nonnegative r.v.-s and therefore we observe incomplete data. In article [1] we consider only right censoring model and problem of estimation of survival function when the survival times and censoring times are dependent and estimates of survival function assuming that the dependence structure is described by a known Archimedean copula function. We demonstrate almost sure asymptotic representation which provides a key tool for obtaining weak convergence result for estimator.

Rerefences

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[3] Sklar A. Fonctions de repartition à n dimensions et leurs marges, Vol.8. of *Publications de l'Institut de Statistique de l'Université de Paris*, 229–231, 1959.