Weak and strong nilpotentizability in the monster towers hosting flag distributions

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Control systems linear in controls, with linearly independent vector field' generators, sometimes happen to be locally nilpotentizable. That is, to locally possess bases that generate (over reals, not over functions) nilpotent algebras of vector fields. The existence of a nilpotent basis may be somehow mischievously hidden in the nature of a system. When it exists and is at hand, a number of key control problems related with the system (e. g., motion planning) become much simpler. We call such systems *weakly* nilpotent. When a system Σ is given globally on a manifold M, we call weakly nilpotent those points in M, around which Σ is weakly nilpotent.
In turn, *strongly* nilpotent are those points p in M, around which Σ is equivalent to its *nilpotent approximation* at p. Naturally, 'strongly' implies 'weakly', but not vice versa: 'strongly' appears to be a much more stringent property.
An important class of weakly nilpotentizable distributions are *Goursat* distributions – members of Goursat *flags* which live on so-called Monster Manifolds ([2]). Local nilpotent bases found for Goursat distributions permit much more – to compute the *nilpotency orders* (sometimes also called 'indices', sometimes 'steps') of the generated real Lie algebras, see [3].
A big problem, with only partial answers known to-date, reads

Problem 1. What points in the Goursat Monster Tower are strongly nilpotent?

In parallel, much ampler classes of globally weakly nilpotent distributions are being furnished by so-called *special m-flags*, $m \geq 2$. Those are induced by rather particular rank-$(m+1)$ subbundles $D \subset TM$, $\dim M = (r+1)m+1$, $r =$ the length of a flag. The defining conditions demand that the tower of consecutive Lie squares of D

$$D \subset [D,\, D] \subset [[D,\, D],\, [D,\, D]] \subset \cdots\cdots \subset TM \tag{1}$$

grow in ranks, at every point of M, in the arithmetical progression $m+1,\, 2m+1,\, 3m+1, \ldots, (r+1)m+1 = \dim M$ and that the associated subtower of Cauchy-characteristic subdistributions $L(D) \subset L([D,\, D]) \subset L([[D,\, D],\, [D,\, D]]) \subset \cdots$ also grow in ranks arithmetically $m,\, 2m,\, 3m,\, \ldots,\, (r-1)m,\, rm$. (The biggest term in this subtower is, strictly speaking, not Cauchy-characteristic, but so-called *covariant subdistribution* of the one before last term in the main tower (1). More on that see, e. g., [1] and [4].)
Much like for Goursat structures, there exist huge manifolds locally universal for the special m-flags of any fixed length r. Upon floating r, one gets a tower of such manifolds. Each member of any special m-flag is locally materialized – up to the local diffeo equivalence – somewhere on certain stage of the tower. This is precisely the mentioned local universality of the tower. All such distributions are globally weakly nilpotent, and relevant local nilpotent bases for them can be effectively constructed. They depend on a natural stratification of germs of special m-flags into so-called *singularity classes*, [5]. The Lie algebras that are generated depend *but* on singularity classes, and the same, obviously, holds for the nilpotency orders. Those orders can by effectively written down (and computed) [4].
In contrast, it is not known (excepting, that is true, many simple cases)

Problem 2. What points in the Special m-Flags' Monster Towers are strongly nilpotent?

Also, barring a few relatively simple situations, it is not known

Problem 3. What are the dimensions of the nilpotent real Lie algebras mentioned in the present abstract?

The nilpotency orders are tractable, but not the real dimensions.

Rerefences

[1] Antonio Kumpera & Jacques Rubin. Multi-flag systems and ordinary differential equations. *Nagoya Mathematical Journal* **166** (2002), 1 – 27.

[2] Richard Montgomery & Michail Zhitomirskii. Points and Curves in the Monster Tower. *Memoirs of the American Mathematical Society* **956** (2010).

[3] Piotr Mormul. Goursat distributions not strongly nilpotent in dimensions not exceeding seven. Volume **281** (2003) of *Lecture Notes in Control and Information Sciences*, 249 – 261.

[4] Piotr Mormul. Multi-dimensional Cartan prolongation and special k-flags. Volume **65** (2004) of *Banach Center Publications*, 157 – 178.

[5] Piotr Mormul. Singularity classes of special 2-flags. *SIGMA* **5** (2009), 102, 22 pages (electronic).