Fractal properties of sets associated with Markov representation of real numbers defined by a double stochastic matrix

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Let $A = \{0, 1\}$ be a number system alphabet, $q = (q_0, q_1)$ be an ordered set of positive numbers, such that $q_0 + q_1 = 1$, $L = A \times A \times \ldots$ be a sequence space, and

$$Q = ||q_{ik}|| = \begin{pmatrix} q_{00} & q_{01} \\ q_{10} & q_{11} \end{pmatrix} = \begin{pmatrix} a & 1-a \\ 1-a & a \end{pmatrix}$$

be a double stochastic matrix (a matrix having nonnegative elements and such that the sum of elements in each row and each column equals 1), where $0 < a < 1$.

Define an interval system of the first rank being a partition of $[0; 1]$:

$$[0; 1] = \Delta_0 \cup \Delta_1, \text{ where } \Delta_0 = [0; q_0)\,; \ \Delta_1 = [q_0; 1]$$

An interval system of the rank n $(n \geqslant 2)$ is defined by the follows conditions:

1. $\Delta_{c_1c_2\ldots c_m} = \Delta_{c_1c_2\ldots c_m0} \cup \Delta_{c_1c_2\ldots c_m1}$;
2. $\min\Delta_{c_1c_2\ldots c_m(i+1)} = \sup\Delta_{c_1c_2\ldots c_mi}$, $i = \{0, 1\}$;
3. $\frac{|\Delta_{c_1\ldots c_mij}|}{|\Delta_{c_1\ldots c_m}|} = q_{ij}$;
4. for any sequence $(c_m) \in L$ the intersection $\bigcap\limits_{m=1}^{\infty} \Delta_{c_1c_2\ldots c_m} = x \equiv \Delta_{c_1c_2\ldots c_m\ldots}$ is a point of $[0; 1]$.

A symbolic representation $\Delta_{c_1c_2\ldots c_m\ldots}$ of a number $x \in [0; 1]$ is called its *Markov representation.*

Theorem 1. *The set*

$$C = \{x : \quad x = \Delta_{c_1c_2\ldots c_n\ldots}, \ c_{2k-1}c_{2k} \in \{00, 11\} \ \forall k \in \mathbb{N}\} \quad \textit{is}$$

the set of Lebesgue measure zero. Its Hausdorff-Besikovitch dimension is a unique solution of the following equation

$$a^x(a^x + (1-a)^x) = 1.$$

Theorem 2. *The set*

$$D = \{x : \quad x = \Delta_{c_1c_2\ldots c_n\ldots}, \ c_k + c_{k+1} + c_{k+2} \neq 1 \ \forall k \in \mathbb{N}\} \quad \textit{is}$$

the set of Lebesgue measure zero. Its Hausdorff-Besikovitch dimension is a unique the solution of the following equation

$$(a(1-a)^2)^x + a^x = 1.$$

Rerefences

[1] Працьовитий М. В. Фрактальний підхід у дослідженнях сингулярних розподілів. — Київ: Вид-во НПУ імені М. П. Драгоманова, 1998. — 296с.