Contractibility of manifolds by means of stochastic flows

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In [2] Xue-Mei Li studied stability of stochastic differential equations and the interplay between the moment stability of a SDE and the topology of the underlying manifold. In particular, she gave sufficient condition on SDE on a manifold \( M \) under which the fundamental group \( \pi_1 M = 0 \). We prove that in fact under essentially weaker conditions the manifold \( M \) is contractible, that is all homotopy groups \( \pi_k M, k \geq 1 \), vanish.

Let \( M \) be a smooth connected manifold (i.e. locally Euclidean Hausdorff topological space with countable base) of dimension \( m \) possibly non-compact and having a boundary and \( \mathcal{F} = (\Omega, \mathcal{F}, \mathbf{P}) \) be a probability space, so \( \Omega \) is a set, \( \mathcal{F} \) is a \( \sigma \)-algebra of subsets of \( \Omega \), and \( \mathbf{P} \) is a probability measure on \( \mathcal{F} \). Let also \( \{\mathcal{F}_t\}_{t \geq 0} \) for some \( \alpha \geq 0 \) be a family of \( \sigma \)-algebras in \( \mathcal{F} \) with the following properties:

- each \( \mathcal{F}_t \) contains all null sets of \( \mathcal{F} \);
- \( \mathcal{F}_s \subseteq \mathcal{F}_t \) for \( s < t \);
- \( \{\mathcal{F}_t\}_{t \geq 0} \) is right continuous in the sense that \( \mathcal{F}_s = \cap_{s < t} \mathcal{F}_t \) for all \( s \geq 0 \).

A map \( \xi : M \times [0, +\infty) \times \Omega \rightarrow M \) will be called a stochastic deformation whenever there exists \( N \in \mathcal{F} \) of measure 0 such that for each \( \omega \in \Omega \setminus N \):

(a) the map \( \xi_{x,t} : \Omega \rightarrow M, \xi_{x,t}(\omega) = \xi(x, t, \omega) \), is \( \mathcal{F}_t / \mathcal{B}(M) \)-measurable;
(b) the map \( \xi_\omega : M \times [0, +\infty) \rightarrow M, \xi_t(x, t) = \xi(x, t, \omega) \), is continuous;
(c) \( \xi(x, 0, \omega) = x \) for all \( x \in M \).

If in addition to (a) and (b) the map \( \xi \) satisfies “semi-group property”:

(d) \( \xi_\omega(\xi_\omega(x, s), t) = \xi_\omega(x, s + t) \) for all \( s, t \geq 0 \),
then \( \xi \) is called an autonomous stochastic flow.

Given a stochastic deformation \( \xi \) one can define the following \( \sigma \)-additive probability measures \( \mu_{x,t}, (x, t) \in M \times [0, +\infty) \) on \( M \) by

\[
\mu_{x,t}(A) := \mathbf{P}\{\omega \in \Omega : \xi(x, t, \omega) \in A\}.
\]

**Theorem 1.** Suppose \( \rho \) is a complete Riemannian metric on \( M \) and \( \xi : M \times [0, +\infty) \times \Omega \rightarrow M \) is a stochastic deformation having the following properties:

(i) the map \( \xi_{t,\omega} : M \rightarrow M, \xi_{t,\omega}(x) = \xi(x, t, \omega) \), is \( C^1 \) for all \( t \in [0, +\infty) \) and \( \omega \in \Omega \setminus N \);
(ii) for each compact subset \( L \) of the tangent bundle \( TM \) we have that

\[
\int_0^\infty \sup_{(x,v) \in L} \mathbf{E} \|T_x \xi_{t,\omega}(v)\| dt < \infty,
\]

where \( \mathbf{E} f = \int_{\Omega} f d\mathbf{P} \) is a mean value, and the norm is taken with respect to \( \rho \);
(iii) there exists a point \( z \in M \), a compact subset \( K \subset M \), \( \varepsilon > 0 \) and \( N > 0 \) such that \( \mu_{z,t}(K) > \varepsilon \) for all \( t > N \).

Then \( M \) is contractible.

**References**
