Contractibility of manifolds by means of stochastic flows

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In [2] Xue-Mei Li studied stability of stochastic differential equations and the interplay between the moment stability of a SDE and the topology of the underlying manifold. In particular, she gave sufficient condition on SDE on a manifold M under which the fundamental group $\pi_1 M = 0$. We prove that in fact under essentially weaker conditions the manifold M is contractible, that is all homotopy groups $\pi_k M$, $k \geq 1$, vanish.

Let M be a smooth connected manifold (i.e. locally Euclidean Hausdorff topological space with countable base) of dimension m possibly non-compact and having a boundary and $\mathcal{T} = (\Omega, \mathcal{F}, \mathbf{P})$ be a probability space, so Ω is a set, $\mathcal{F}$ is a σ-algebra of subsets of Ω, and $\mathbf{P}$ is a probability measure on $\mathcal{F}$. Let also $\{\mathcal{F}_t\}_{t\geq 0}$ for some $a \geq 0$ be a family of σ-algebras in $\mathcal{F}$ with the following properties:

- each $\mathcal{F}_t$ contains all null sets of $\mathcal{F}$;
- $\mathcal{F}_s \subseteq \mathcal{F}_t$ for $s < t$;
- $\{\mathcal{F}_t\}_{t\geq 0}$ is right continuous in the sense that $\mathcal{F}_s = \cap_{s<t} \mathcal{F}_t$ for all $s \geq 0$.

A map $\xi : M \times [0, +\infty) \times \Omega \to M$ will be called a *stochastic deformation* whenever there exists $N \in \mathcal{F}$ of measure 0 such that for each $\omega \in \Omega \setminus N$:

(a) the map $\xi_{x,t} : \Omega \to M$, $\xi_{x,t}(\omega) = \xi(x, t, \omega)$, is $\mathcal{F}_t/\mathcal{B}(M)$-measurable;
(b) the map $\xi_\omega : M \times [0, +\infty) \to M$, $\xi_t(x, t) = \xi(x, t, \omega)$, is continuous;
(c) $\xi(x, 0, \omega) = x$ for all $x \in M$.

If in addition to (a) and (b) the map ξ satisfies "semi-group property":

(d) $\xi_\omega(\xi_\omega(x, s), t) = \xi_\omega(x, s + t)$ for all $s, t \geq 0$,

then ξ is called an *autonomous stochastic flow.*

Given a stochastic deformation ξ one can define the following σ-additive probability measures $\mu_{x,t}$, $(x, t) \in M \times [0, +\infty)$ on M by

$$\mu_{x,t}(A) := \mathbf{P}\{\omega \in \Omega : \xi(x, t, \omega) \in A\}.$$

Theorem 1. *Suppose ρ is a complete Riemannian metric on M and $\xi : M \times [0, +\infty) \times \Omega \to M$ is a stochastic deformation having the following properties:*

(i) *the map $\xi_{t,\omega} : M \to M$, $\xi_{t,\omega}(x) = \xi(x, t, \omega)$, is C^1 for all $t \in [0, +\infty)$ and $\omega \in \Omega \setminus N$;*
(ii) *for each compact subset $\mathbf{L}$ of the tangent bundle TM we have that*

$$\int_0^{+\infty} \sup_{(x,v)\in\mathbf{L}:\ x\in M,\ v\in T_xM} \mathbf{E}\, \|T_x\xi_{t,\omega}(v)\| dt < \infty,$$

where $\mathbf{E}f = \int_\Omega f d\mathbf{P}$ is a mean value, and the norm is taken with respect to ρ;
(iii) *there exists a point $z \in M$, a compact subset $K \subset M$, $\varepsilon > 0$ and $N > 0$ such that $\mu_{z,t}(K) > \varepsilon$ for all $t > N$.*

Then M is contractible.

Rerefences

[1] Alexandra Antoniouk, Sergiy Maksymenko, *Contractibility of manifolds by means of stochastic flows*, arXiv:1111.0070v3, 2017

[2] X.-M. Li, *Stochastic differential equations on noncompact manifolds: moment stability and its topological consequences*, Probab. Theory Related Fields 100(4) (1994) 417–428.