Moyal and Rankin-Cohen deformations of algebras

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Denote $\mathbb{H} = \{z \in \mathbb{C} \mid \mathrm{Im} z > 0\}$, $\mathbb{C}^\times = \mathbb{C} - \{0\}$. The left action

$$SL(2,\mathbb{R}) \times (\mathbb{H} \times \mathbb{C}^\times) \to \mathbb{H} \times \mathbb{C}^\times, \qquad \gamma(z,X) = \left(\frac{az+b}{cz+d}, \frac{X}{cz+d} \right),$$

where $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, induces the right action of $SL(2,\mathbb{R})$ on the algebra of holomorphic functions $\mathcal{H}ol(\mathbb{H} \times \mathbb{C}^\times)$. The restriction of this action to the subspace $X^k \mathcal{H}ol(\mathbb{H})$, $k \in \mathbb{Z}$, equips $\mathcal{H}ol(\mathbb{H})$ with the right action $|_k$ of $SL(2,\mathbb{R})$. These actions are important in the definition of modular (automorphic) forms [4]. The Rankin–Cohen brackets on this subalgebra were defined in [1] as follows. Let $k, l \in \mathbb{Z}$, $n \in \mathbb{N} = \mathbb{Z}_{\geqslant 0}$, $f, g \in \mathcal{H}ol(\mathbb{H})$. Then

$$[X^k f, X^l g]_n^{RC} = X^{k+l+2n} \sum_{r+s=n}^{r,s \geqslant 0} (-1)^s \binom{k+n-1}{s} \binom{l+n-1}{r} f^{(r)} g^{(s)},$$

where $f^{(r)} = \partial_z^r f$. Cohen proved in [1, Theorem 7.1] that the operation $[\_,\_]_n^{RC}$ on $\mathcal{H}ol(\mathbb{H})[X^{-1}, X]$ is $SL(2,\mathbb{R})$-equivariant. A *deformation* of an associative $\mathbb{C}$-algebra $(A, m : A \otimes A \to A)$ is a $\mathbb{C}$-bilinear map $\mu : A \times A \to A[[\hbar]]$, $\mu(a,b) = \sum_{n=0}^\infty \hbar^n \mu_n(a,b)$ such that

$$\mu_0(a,b) = m(a \otimes b) = ab, \qquad \mu(1,a) = a = \mu(a,1)$$

and the $\mathbb{C}[[\hbar]]$-bilinear map $\tilde{\mu} : A[[\hbar]] \times A[[\hbar]] \to A[[\hbar]]$, which extends μ by $\mathbb{C}[\hbar]$-bilinearity and $\hbar$-adic continuity, is associative. Let $\mathfrak{a} \subset \mathrm{Der} A$ be an abelian $\mathbb{C}$-subalgebra of the Lie $\mathbb{C}$-algebra of derivations of A. View an element $P = \sum_{i=1}^l \xi_i \otimes \eta_i \in \mathfrak{a} \otimes_\mathbb{C} \mathfrak{a}$ as a $\mathbb{C}$-linear operator $A^{\otimes 2} \to A^{\otimes 2}$. It is well-known that

$$a \star_P b \equiv \mu_P(a,b) = m \exp(\hbar P).(a \otimes b) = \sum_{n=0}^\infty \frac{\hbar^n}{n!} P^n.(a \otimes b)$$

is a deformation of A. An example is provided by the Moyal deformation of $A = \mathcal{H}ol(\mathbb{C}^2)$, $\mathbb{C}^2 = \{(q,p)\}$, $P = M = \partial_p \otimes \partial_q - \partial_q \otimes \partial_p$. The point of this note is to show that the Rankin–Cohen brackets give a deformation, related to the Moyal deformation. This is already proven by V. Ovsienko [3] using different formulas and in a different way.

Consider the embedding

$$\Psi = \left(\mathbb{H} \times \mathbb{C}^\times \hookrightarrow \mathbb{C} \times \mathbb{C}^\times \xrightarrow[\cong]{\psi} \mathbb{C} \times \mathbb{C}^\times \hookrightarrow \mathbb{C}^2 \right)$$

where $\psi(z,X) = (zX^{-1}, X^{-1}) = (q,p)$. Vector field ∂_p (resp. ∂_q) on $\mathbb{C}^2$ is lifted along Ψ to vector field $\xi = -X^2 \partial_X - zX\partial_z$ (resp. $\eta = X\partial_z$) on $\mathbb{H} \times \mathbb{C}^\times$. Thus, $M = \partial_p \otimes \partial_q - \partial_q \otimes \partial_p$ is lifted to $P = RC = \xi \otimes \eta - \eta \otimes \xi$ and ξ and η commute. Hence,

$$F \star_{RC} G = m \exp(\hbar RC).(F \otimes G) = \sum_{n=0}^\infty \frac{\hbar^n}{n!} m(RC)^n.(F \otimes G)$$

is a deformation of $\mathcal{H}ol(\mathbb{H} \times \mathbb{C}^\times)$.

Theorem 1. *Let $k, l \in \mathbb{Z}$, $n \in \mathbb{N}$. Assume that $k > 0$ or $l > 0$ or $k, l \leqslant 0$, $n < \max\{1-k, 1-l\}$ or $k, l \leqslant 0$, $n > 1-k-l$. Then for all $f, g \in \mathcal{H}ol(\mathbb{H})$ and $F = X^k f, G = X^l g$ we have*

$$(n!)^{-1} m(RC)^n.(F \otimes G) = [F,G]_n^{RC}.$$

Proof. The embedding $\Psi : \mathbb{H} \times \mathbb{C}^\times \hookrightarrow \mathbb{C}^2$ is $SL(2,\mathbb{R})$-equivariant, where the left action on $\mathbb{C}^2 = \{(q,p)\}$ is the standard action of matrices on vectors. The mapping

$$M = \partial_p \otimes \partial_q - \partial_q \otimes \partial_p \in \mathrm{End}_{\mathbb{C}}(\mathcal{H}ol(\mathbb{C}^2) \otimes \mathcal{H}ol(\mathbb{C}^2))$$

commutes with the action of $SL(2,\mathbb{R})$, or, equivalently, of the Lie algebra $\mathfrak{sl}(2,\mathbb{R})$. Hence, the Moyal deformation is $SL(2,\mathbb{R})$-equivariant. This implies and can be checked directly that

$$RC = \xi \otimes \eta - \eta \otimes \xi \in \mathrm{End}_{\mathbb{C}}(\mathcal{H}ol(\mathbb{H} \times \mathbb{C}^\times) \otimes \mathcal{H}ol(\mathbb{H} \times \mathbb{C}^\times))$$

commutes with the action of $\mathfrak{sl}(2,\mathbb{R})$ and $SL(2,\mathbb{R})$. Thus,

$$\star_{RC} : \mathcal{H}ol(\mathbb{H} \times \mathbb{C}^\times) \otimes \mathcal{H}ol(\mathbb{H} \times \mathbb{C}^\times) \to \mathcal{H}ol(\mathbb{H} \times \mathbb{C}^\times)[[\hbar]]$$

is a homomorphism of $SL(2,\mathbb{R})$-modules. Accordingly to El Gradechi [2, Proposition 3.11] if the hypotheses on (k,l,n) hold, then there is only 1-dimensional vector space of $SL(2,\mathbb{R})$-equivariant bidifferential operators

$$(\mathcal{H}ol(\mathbb{H}), |_k) \otimes (\mathcal{H}ol(\mathbb{H}), |_l) \to (\mathcal{H}ol(\mathbb{H}), |_{k+l+2n}).$$

Both maps $f \otimes g \mapsto (n!)^{-1} m(RC)^n.(X^k f \otimes X^l g)$ and $f \otimes g \mapsto [X^k f, X^l g]_n^{RC}$ belong to this space, therefore, they are proportional. One checks that the proportionality constant is 1. In fact, evaluate both operators on the following $f \otimes g$. If $k = l = n = 0$ take $f = g = 1$. If $k > 0$ or $k < 0$, $k \leqslant l \leqslant 0$, $n < 1 - k$ take $f = 1$, $g = z^n$. If $l > 0$ or $l < 0$, $l \leqslant k \leqslant 0$, $n < 1 - l$ take $f = z^n$, $g = 1$. If $k, l \leqslant 0$ and $n > 1 - k - l$ choose $r, s \in \mathbb{N}$, such that $r + s = n$, $r + k > 0$ and $s + l > 0$, and take $f(z) = z^r$ and $g(z) = z^s$ (use coordinates (q,p) to make the computations easier). □

Corollary 2. *The Rankin–Cohen deformation $\star_{RC}$ restricted to $A^{\otimes 2}$, $A = \mathcal{H}ol(\mathbb{H})[X]$ coincides with the map $A \otimes A \to A[[\hbar]]$, $X^k f \otimes X^l g \mapsto \sum_{n=0}^{\infty} \hbar^n [X^k f, X^l g]_n^{RC}$.*

Proof. For all $(k,l,n) \in \mathbb{N}^3$ the statement follows from Theorem 1 except for $(k,l,n) = (0,0,1)$. In the latter case $[f,g]_1 = 0 = m(RC)(f \otimes g)$ for all $f, g \in \mathcal{H}ol(\mathbb{H})$. □

Rerefences

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