Constructive description of G-monogenic mappings in the algebra of complex quaternions

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Let $\mathbb{H}(\mathbb{C})$ be the quaternion algebra over the field of complex numbers $\mathbb{C}$, whose basis consists of the unit 1 of the algebra and of the elements I, J, K satisfying the multiplication rules:

$$I^2 = J^2 = K^2 = -1,$$

$$IJ = -JI = K, \quad JK = -KJ = I, \quad KI = -IK = J.$$

In the algebra $\mathbb{H}(\mathbb{C})$ there exists another basis $\{e_1, e_2, e_3, e_4\}$ such that multiplication table in a new basis can be represented as

$\cdot$	e_1	e_2	e_3	e_4
e_1	e_1	0	e_3	0
e_2	0	e_2	0	e_4
e_3	0	e_3	0	e_1
e_4	e_4	0	e_2	0

.

The unit of the algebra can be decomposed as $1 = e_1 + e_2$.

Let us consider the vectors

$$i_1 = e_1 + e_2, \quad i_2 = a_1 e_1 + a_2 e_2, \quad i_3 = b_1 e_1 + b_2 e_2, \tag{1}$$

where $a_k, b_k \in \mathbb{C}$, $k = 1, 2$, which are linearly independent over the field of real numbers $\mathbb{R}$. It means that the equality $\alpha_1 i_1 + \alpha_2 i_2 + \alpha_3 i_3 = 0$ for $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$ holds if and only if $\alpha_1 = \alpha_2 = \alpha_3 = 0$.

In the algebra $\mathbb{H}(\mathbb{C})$ we consider the linear span

$$E_3 := \{\zeta = x i_1 + y i_2 + z i_3 : x, y, z \in \mathbb{R}\}$$

generated by the vectors i_1, i_2, i_3 over the field $\mathbb{R}$.

In the paper [1] we introduced a new class of quaternionic mappings, so-called, G-monogenic mappings.

A continuous mapping $\Phi : \Omega_\zeta \to \mathbb{H}(\mathbb{C})$ $\big($or $\widehat{\Phi} : \Omega_\zeta \to \mathbb{H}(\mathbb{C})\big)$ is *right-G-monogenic* $\big($or *left-G-monogenic*$\big)$ in a domain $\Omega_\zeta \subset E_3$, if Φ $\big($or $\widehat{\Phi}\big)$ is differentiable in the sense of the Gâteaux at every point of Ω_ζ, i. e. for every $\zeta \in \Omega_\zeta$ there exists an element $\Phi'(\zeta) \in \mathbb{H}(\mathbb{C})$ $\big($or $\widehat{\Phi}'(\zeta) \in \mathbb{H}(\mathbb{C})\big)$ such that

$$\lim_{\varepsilon \to 0+0} \Big(\Phi(\zeta + \varepsilon h) - \Phi(\zeta)\Big)\varepsilon^{-1} = h\Phi'(\zeta) \quad \forall\, h \in E_3$$

$$\left(\text{or} \quad \lim_{\varepsilon \to 0+0} \Big(\widehat{\Phi}(\zeta + \varepsilon h) - \widehat{\Phi}(\zeta)\Big)\varepsilon^{-1} = \widehat{\Phi}'(\zeta)h \quad \forall\, h \in E_3\right).$$

We introduce linear functionals $f_1 : \mathbb{H}(\mathbb{C}) \to \mathbb{C}$ and $f_2 : \mathbb{H}(\mathbb{C}) \to \mathbb{C}$ by setting

$$f_1(e_1) = f_1(e_3) = 1, \qquad f_1(e_2) = f_1(e_4) = 0,$$

$$f_2(e_2) = f_2(e_4) = 1, \qquad f_2(e_1) = f_2(e_3) = 0.$$

Denote by $f_k(E_3) := \{f_k(\zeta) : \zeta \in E_3\}$ for $k = 1, 2$.

Note that the points $(x, y, z) \in \mathbb{R}^3$ corresponding to the non-invertible elements $\zeta = x i_1 + y i_2 + z i_3 \in E_3$ lie on the straight lines

$$L^1 :\ x + y\mathrm{Re}\, a_1 + z\mathrm{Re}\, b_1 = 0, \qquad y\mathrm{Im}\, a_1 + z\mathrm{Im}\, b_1 = 0,$$

$$L^2: \; x + y\mathrm{Re}\, a_2 + z\mathrm{Re}\, b_2 = 0, \qquad y\mathrm{Im}\, a_2 + z\mathrm{Im}\, b_2 = 0$$

in the three-dimensional space $\mathbb{R}^3$.

Denote by

$$D_1 := f_1(\Omega_\zeta) \subset \mathbb{C}, \qquad D_2 := f_2(\Omega_\zeta) \subset \mathbb{C}.$$

In the following theorems we established constructive description of all G-monogenic mappings using four analytic functions of the complex variable.

Theorem 1. *Let a domain Ω be convex in the direction of the straight lines L^1 and L^2 and let $f_1(E_3) = f_2(E_3) = \mathbb{C}$. Then every right-$G$-monogenic mapping $\Phi : \Omega_\zeta \to \mathbb{H}(\mathbb{C})$ has the form*

$$\Phi(\zeta) = F_1(\xi_1)e_1 + F_2(\xi_2)e_2 + F_3(\xi_1)e_3 + F_4(\xi_2)e_4,$$

where F_1 and F_3 are functions of the variable $\xi_1 := x + ya_1 + zb_1$ analytic in the domain D_1, and F_2 and F_4 are functions of the variable $\xi_2 := x + ya_2 + zb_2$ analytic in the domain D_2.

Theorem 2. *Let a domain Ω be convex in the direction of the straight lines L^1 and L^2 and let $f_1(E_3) = f_2(E_3) = \mathbb{C}$. Then every left-$G$-monogenic mapping $\widehat{\Phi} : \Omega_\zeta \to \mathbb{H}(\mathbb{C})$ has the form*

$$\widehat{\Phi}(\zeta) = \widehat{F}_1(\xi_1)e_1 + \widehat{F}_2(\xi_2)e_2 + \widehat{F}_3(\xi_2)e_3 + \widehat{F}_4(\xi_1)e_4,$$

where $\widehat{F}_1$ and $\widehat{F}_4$ are functions of the variable $\xi_1 := x + ya_1 + zb_1$ analytic in the domain D_1, and $\widehat{F}_2$ and $\widehat{F}_3$ are functions of the variable $\xi_2 := x + ya_2 + zb_2$ analytic in the domain D_2.

Rerefences

[1] V. S. Shpakivskyi, T. S. Kuzmenko. On one class of quaternionic mappings. *Ukr. Math. J.*, 68(1) : 127 – 143, 2016.