Orientations of trees and signed Markov graphs

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For every vertex map $\sigma : V(X) \to V(X)$ on a finite tree X one can construct its *Markov graph* $\Gamma(X, \sigma)$ which is a digraph that encodes σ-covering relation between edges in X. By definition, $\Gamma(X, \sigma)$ has a vertex set equals the edge set $E(X)$ of X and for every edge $uv \in E(X)$ its out-neighbourhood in $\Gamma(X, \sigma)$ equals the edge set $E([\sigma(u), \sigma(v)]_X)$ of a unique shortest $\sigma(u) - \sigma(v)$ path in X.

Let $\tau : E(X) \to V(X)$ be an orientation of a tree X. For each non-constant map $\sigma : V(X) \to V(X)$ the orientation τ defines a map $s_\tau : A(\Gamma(X, \sigma)) \to \{1, -1\}$ in such a way that $s_\tau(e_1, e_2) = 1$, if $pr_{\,e_2}(\sigma(\tau(e_1))) = \tau(e_2)$ and $s_\tau(e_1, e_2) = -1$, otherwise. The pair $\Gamma^\tau(X, \sigma) = (\Gamma(X, \sigma), s_\tau)$ is the *signed Markov graph* of σ. Denote by $M_{\Gamma^\tau(X,\sigma)}$ the adjacency matrix of $\Gamma^\tau(X, \sigma)$. If σ is a constant map, then by definition $M_{\Gamma^\tau(X,\sigma)}$ is a null matrix. It is well-known (see [1, 2]) that for any fixed orientation τ of X the correspondence $\sigma \to M_{\Gamma^\tau(X,\sigma)}$ establishes a homomorphism from the full transformation semigroup T_n to the matrix semigroup $Mat_{\,n-1}(\mathbb{Z})$. Note that this correspondence is "almost" injective in the sense that $M_{\Gamma^\tau(X,\sigma_1)} = M_{\Gamma^\tau(X,\sigma_2)}$ if and only if $\sigma_1 = \sigma_2$ or σ_1 and σ_2 are both constant maps.

Theorem 1. [3] *For any orientation τ and a map σ the trace of $M_{\Gamma^\tau(X,\sigma)}$ equals $|fix\,\sigma| - 1$, where $fix\,\sigma$ is the set of σ-fixed points.*

A map σ is called *τ-positive* provided $s_\tau \equiv 1$. Similarly, one can define *τ-negative* maps. By definition constant maps are τ-positive and τ-negative for all orientations τ.

Proposition 2. *A map σ is τ-positive for all τ if and only if σ is a projection on some connected set of vertices. Similarly, σ is τ-positive for all τ if and only if σ is constant.*

For a map σ an edge $uv \in E(X)$ is called *σ-positive* (*σ-negative*) if $pr_{\,uv}(\sigma(u)) = u$ and $pr_{\,uv}(\sigma(v)) = v$ ($pr_{\,uv}(\sigma(u)) = v$ and $pr_{\,uv}(\sigma(v)) = u$). Denote by $p(X, \sigma)$ and $n(X, \sigma)$ the number of σ-positive and σ-negative edges in X, respectively.

Proposition 3. *If a map σ is τ-positive (τ-negative) for some τ, then $n(X, \sigma) = 0$ ($p(X, \sigma) = 0$).*

A map σ is called *metric* if $d_X(\sigma(u), \sigma(v)) \le d_X(u, v)$ for all pairs of vertices $u, v \in V(X)$. It is easy to see that σ is metric if and only if $[\sigma(u), \sigma(v)]_X \subset \sigma([u, v]_X)$ for all $u, v \in V(X)$. A map σ is called *linear* provided $\sigma([u, v]_X) \subset [\sigma(u), \sigma(v)]_X$ for all $u, v \in V(X)$.

Proposition 4. *Let σ be a metric or a linear map. Then $n(X, \sigma) \le 1$. Moreover, the equality $n(X, \sigma) = 1$ implies $p(X, \sigma) = 0$.*

Theorem 5. *Let σ be a metric or a linear map. If $n(X, \sigma) = 0$, then there exists an orientation τ such that σ is τ-positive. Similarly, if $n(X, \sigma) = 1$ (and thus $p(X, \sigma) = 0$), then there is τ such that σ is τ-negative.*

Rerefences

[1] Chris Bernhardt. Vertex maps for trees: algebra and periods of periodic orbits. *Discrete Contin. Dyn. Syst.*, 14(3) : 399–408, 2006.

[2] Bau-Sen Du. On the class of similar square $\{-1, 0, 1\}$-matrices arising from vertex maps on trees. Preprint, arXiv:1503.04568, 2015.

[3] Sergiy Kozerenko. Discrete Markov graphs: loops, fixed points and maps preordering. *J. Adv. Math. Stud.*, 9(1) : 99–109, 2016.