On projective classes of rational functions

Konovenko Nadiia
(Department of Mathematics, ONAFT, Odessa, Ukraine)
E-mail: `konovenko@ukr.net`

Lychagin Valentin
(Department of Mathematics, University of Tromsö, Norway)
E-mail: `valentin.lychagin@uit.no`

We study equivalence classes of rational functions on Riemann's sphere $\overline{\mathbb{C}}$ with respect to Möbius transformations (cf. [1]). The Lie group $\mathbf{SL}_2(\mathbb{C})$ acts on $\overline{\mathbb{C}}$ and the corresponding representations of Lie algebra $\mathbf{sl}_2(\mathbb{C})$ is given by the vector fields

$$\mathbf{sl}_2(\mathbb{C}) = \langle \frac{\partial}{\partial z}, \quad z\frac{\partial}{\partial z}, \quad z^2\frac{\partial}{\partial z} \rangle.$$

Denote by J^k the manifold of k-jets of analytical functions on $\overline{\mathbb{C}}$, and let $(z, u, u_1, \ldots, u_k)$ be the natural (local) coordinates on J^k.

Then a rational function I on the manifold J^k we call a projective invariant of order $\leq k$, if $X^{(k)}(I) = 0$, for all vector fields $X \in \mathbf{sl}_2(\mathbb{C})$. Here we denoted by $X^{(k)}$ the k-th prolongations of the vector field X.

Theorem 1. *The field of rational differential invariants of order $\leq k$ is generated by invariants*

$$J_0 = u, \qquad J_3 = u_1^{-3}u_3 - \frac{3}{2}u_2^2u_1^{-4},$$

and invariant derivatives

$$J_4 = \nabla(J_3), \quad \ldots, \quad J_k = \nabla^{k-3}(J_3),$$

where $\nabla = u_1^{-1}\frac{d}{dz}$.

This field separates regular $\mathbf{SL}_2(\mathbb{C})$ orbit.

For given rational function $f = f(z)$ we denote $J_3(f)$ the value of the projective invariant J_3 on f.

The transcendent degree of the field of rational functions on the Riemann sphere equals one, and therefore between functions f and $J_3(f)$ there is an algebraic relation

$$R\big(f, J_3(f)\big) = 0. \tag{1}$$

The polynomial R_f, which satisfies (1) and having the smallest degree, we call generating polynomial for f.

Theorem 2. (1) *The projective class of a rational function $f(z)$ is defined by the generating polynomial $R_f(f, J_3(f))$.*

(2) *Two rational functions $f(z)$, $g(z)$ are projectively equivalent if and only if the function $g(z)$ is a solution of the following ordinary differential equation:*

$$R_f\left(u, u_1^{-3}u_3 - \frac{3}{2}u_2^2u_1^{-4}\right) = 0. \tag{2}$$

Remark that this differential equation is so-called automorphic equation in the sense that the Möbius group $\mathbf{PSL}_2(\mathbb{C})$ acts in a transitive way on the solution space of differential equation (2).

In particular, all solutions of differential equation (2) are rational functions.

Example 3. (1) Solutions of the following differential equation

$$\big(D + 4au\big)^2\big(2u_1u_3 - 3u_2^2\big) + 12a^2u_1^4 = 0$$

are rational functions that are projectively equivalent to polynomials of the second degree $f(z) = az^2 + bz + c$ with given discriminant D and leading coefficient a.

Remark that $\frac{D}{a}$ is a projective invariant.

(2) Solutions of differential equation

$$(u^2 - 4)(2u_1u_3 - 3u_2^2) + 12u_1^4 = 0$$

are rational functions which are projectively invariant to the Jukowski function $f(z) = z + z^{-1}$.

Rerefences

[1] N. Konovenko. Differential invariants and $\mathfrak{sl}_2$ - geometries. *Kiev: "Naukova Dumka" NAS of Ukraine*, 2013, 192 p.