The extremal problem for the area of an image of a disc

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Let Γ be a family of curves γ in the complex plane $\mathbb{C}$. A Borel function $\varrho : \mathbb{C} \to [0,\infty]$ is called *admissible* for Γ, abbr. $\varrho \in \operatorname{adm}\Gamma$, if

$$\int\limits_{\gamma} \varrho(z)\,|dz| \ \geqslant\ 1$$

for all $\gamma \in \Gamma$. The *p–modulus* of Γ is the quantity defined by

$$\mathcal{M}_p(\Gamma) \ =\ \inf_{\varrho\in\operatorname{adm}\Gamma} \int\limits_{\mathbb{C}} \varrho^p(z)\,dx\,dy\,, \quad p \geqslant 1.$$

Let E, F and G be arbitrary sets in $\mathbb{C}$. Denote by $\Delta(E,F,G)$ a family of all continuous curves $\gamma : [a,b] \to \mathbb{C}$ joining E and F in G, i.e. $\gamma(a) \in E$, $\gamma(b) \in F$ and $\gamma(t) \in G$ for $a < t < b$. Given a domain $D \subset \mathbb{C}$ and $z_0 \in D$, denote by $\mathbb{A}(z_0,r_1,r_2) = \{z \in \mathbb{C} : r_1 \leqslant |z-z_0| \leqslant r_2\} < d_0$, where $d_0 = \operatorname{dist}(z_0,\partial D)$.

Now let $Q : D \to [0,\infty]$ be a (Lebesgue) measurable function. A homeomorphism $f : D \to \mathbb{C}$ is called *a ring Q-homeomorphism with respect to p-modulus* at $z_0 \in D$, if

$$\mathcal{M}_p(\Delta(fS_1, fS_2, fD)) \ \leqslant\ \int\limits_{\mathbb{A}} Q(z)\,\eta^p(|z-z_0|)\,dx\,dy$$

for every ring $\mathbb{A} = \mathbb{A}(z_0,r_1,r_2)\,,\ 0 < r_1 < r_2 < d_0$, and every measurable function $\eta : (r_1,r_2) \to [0,\infty]$ such that $\int\limits_{r_1}^{r_2} \eta(r)\,dr \geqslant 1\,.$

Let $\mathbb{B} = \{z \in \mathbb{C} :\ |z| \leqslant 1\}$ and $Q : \mathbb{B} \to [0,\infty]$ be a (Lebesgue) measurable function. For $p > 2$ denote by $\mathcal{H}$ a set of all ring Q-homepmorphisms $f : \mathbb{B} \to \mathbb{C}$ with respect to p-modulus at the origin satisfying

$$q(t) = \frac{1}{2\pi t} \int\limits_{S_t} Q(z)\,|dz| \leqslant q_0\,t^{-\alpha},\ q_0 \in (0,\infty)\,,\ \alpha \in [0,\infty)\,,$$

for almost all $t \in (0,1)$. Here $S_t = \{z \in \mathbb{C} : |z| = t\}$. Let $\mathbf{S}_r(f) = |fB_r|$ be an area functional over the class $\mathcal{H}$ where $B_r = \{z \in \mathbb{C} :\ |z| \leqslant r\}$. The following statement provides an extremal bound for the functional $\mathbf{S}_r(f)$.

Theorem 1. *For all* $r \in [0,1]$

$$\min_{f\in\mathcal{H}} \mathbf{S}_r(f) = \pi\ \left(\frac{p-2}{\alpha+p-2}\right)^{\frac{2(p-1)}{p-2}}\ q_0^{\frac{2}{2-p}}\ r^{\frac{2(\alpha+p-2)}{p-2}}\ .$$