# Deformation of a Morse function on a surface with the boundary

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Let $M$ be a compact smooth connected oriented surface with the boundary $\partial M$, $f : M \to \mathbb{R}$ be a smooth function, which has one critical point on each critical level (simple function), $CP(f)$ ($NDCP(f)$) be the set of (non-degenerated) critical points of the function $f$. We consider the class $\Omega(M)$ of functions $f$ such that the following conditions hold:

- if the critical point $p_0$ of $f$ belongs to the boundary $\partial M$, then $p_0$ is non-degenerated critical point of $f$;
- if the critical point $p_0$ of $f$ is internal ($p_0 \notin \partial M$), then it is non-degenerated critical point of $f$ and $f|_{\partial M}$;
- each critical point $p_0$ of $f|_{\partial M}$ is also critical point of $f$.

In other words, $\Omega(M) = \{f : M \to \mathbb{R} | CP(f) = NDCP(f) = CP(f|_{\partial M}) = NDCP(f|_{\partial M})\}$.

If $p_0 \in CP(f|_{\partial M})$, then $p_0 \in NDCP(f|_{\partial M})$ and $p_0 \in NDCP(f)$.

Let $f \in \Omega(M)$. The components of level lines of function $f$ is called by layer. These layers are homeomorphic to the line segment or to the circle for the regular levels. Then the surface $M$ is divided to the union of the layers which leads to the foliation with the singularities. We call the layer by the layer of the first (the second) type if it corresponds to the component of level line which is homeomorphic to the line segment (circle). Let us consider the equivalence relation on $M$, such that points are equivalent if and only if they belong to one and the same layer. Finally, the consideration of the quotient topology in the space of these layers advances to some graph $\Gamma_f$ with black (for the layers of the first type) and red (for the layers of the second type) edges.

**Definition 1.** A vertex of the graph $\Gamma_f$ of the function $f$ with the valency 3 (4), incident to three black edges, is called by the $\mathsf{Y}$–*vertex* ($\mathsf{X}$–*vertex*).

We fix the cyclic order of the edges in the $\mathsf{Y}$ and $\mathsf{X}$–vertices of the graph $\Gamma_f$ accordingly to the orientation of the surface $M$.

**Definition 2.** An *equipped Kronrod–Reeb graph* of the function $f$ of the *class* $\Omega(M)$ is defined to be the graph $\Gamma_f$ with the following elements:

- black and red edges;
- fixed cycle order in the edges, which are incident to $\mathsf{Y}$ and $\mathsf{X}$–vertices;
- fixed orientation of the edges (upward).

Remind that two given smooth functions $f \in \Omega(M)$ and $g \in \Omega(N)$ are said to be *fiber equipped equivalent* if there exists a homeomorphism $\lambda : M \to N$, which maps the components of the level sets of $f$ onto the components of the level sets of $g$ and preserve the growing directions of functions.

**Definition 3.** Equipped Kronrod–Reeb graphs $\Gamma_f$ and $\Gamma_g$ of the functions $f, g \in \Omega(M)$ are said to be *equivalent* with the isomorphism $\varphi : \Gamma_f \to \Gamma_g$ ($\Gamma_f \sim_\varphi \Gamma_g$) if $\varphi$ satisfies the following conditions:

- preserve the coloration of the edges;
- either preserve the cyclic order in all $\mathsf{Y}$ and $\mathsf{X}$–vertices or simultaneously changes the order in all $\mathsf{Y}$ and $\mathsf{X}$–vertices;

- preserve the orientation of the edges.

**Theorem 4.** *Let $M, N$ be smooth compact surfaces (with the boundaries), $f \in \Omega(M)$, $g \in \Omega(N)$. Then $f$ and $g$ are fiber equipped equivalent if and only if their equipped Kronrod–Reeb graphs $\Gamma_f$ and $\Gamma_g$ are equivalent.*

**Definition 5.** A *simple deformation of a function $f_1 \in \Omega(M)$ into the function $f_2 \in \Omega(M)$* is defined to be a continuous function $F_t(x) := F(x,t) : M \to \mathbb{R}$, such that: $F_0(x) = f_1(x), x \in M$ and $F_1(x) = f_2(x), x \in M$. If there exists the one–point subset $I = \{t_0\} \subset (0,1)$, such that $F_t \in \Omega(M)$, $t \in [0,1]\backslash\{t_0\}$ and $F_{t_0} \notin \Omega(M)$, then we say that the deformation $F_t$ has the *simple catastrophe* in the point $t_0$.

In the case $\Gamma_f$ is a planar graph its correct to consider the function $\varphi$, which puts into the correspondence to each vertex $v_i$ the set of all incident to it edges $e_i^v$, (for the black edges) $l_i^v$ (for the red ones) and two comparison relations: $v_1 \prec v_2$ ($e_1 \prec e_2$) if $v_1$ ($e_1$) is under the $v_2$ ($e_2$) and $e_1 \vdash e_2$ if $e_1$ is left of the edge $e_2$. Then the function $\varphi$, relations $\prec, \vdash$ and the designation, which the edge is the common to the vertices $v_1$ and $v_2$, clearly define the graph $\Gamma_f$.

**Definition 6.** *A simple deformation of Kronrod–Reeb graph* is defined to be one of the following operations or converse to it (let $v_1, v_2$ be adjacent vertices, $v_1 \prec v_2$):

(1) contraction of the vertex $v_2$ and the edge $l_1^{v_2}$, if $\varphi(v_1) = \{e_1^{v_1} \prec l_1^{v_1} = l_1^{v_2}\}$, $\varphi(v_2) = \{l_1^{v_2}\}$;
(2) contraction of the edge $e_1^{v_2} = e_2^{v_1}$ and incident to it vertices $v_1, v_2$ if $\varphi(v_1) = \{e_1^{v_1} \prec e_2^{v_1} = e_1^{v_2} \vdash e_3^{v_1}\}$, $\varphi(v_2) = \{e_1^{v_2}\}$;
(3) contraction of the edge $l_1^{v_2} = l_2^{v_1}$ and incident to it vertices $v_1, v_2$ if $\varphi(v_2) = \{l_1^{v_2}\}$, $\varphi(v_1) = \{l_1^{v_1} \prec l_2^{v_1} = l_1^{v_2} \vdash l_3^{v_1}\}$;
(4) symmetry of the edge $e_3^{v_1}$ if $\varphi(v_1) = \{e_1^{v_1} \prec e_2^{v_1} = e_1^{v_2} \vdash e_3^{v_1}\}$, $\varphi(v_2) = \{e_1^{v_2} \prec e_2^{v_2} \vdash e_3^{v_2}\}$, ie the isomorphism $\delta$, such that $\varphi(\delta(v_1)) = \{e_1^{\delta(v_1)} \prec e_2^{\delta(v_1)} \vdash e_3^{\delta(v_1)} = e_1^{\delta(v_2)}\}$, $\varphi(\delta(v_2)) = \{e_1^{\delta(v_2)} \prec e_2^{\delta(v_2)} \vdash e_3^{\delta(v_2)}\}$;
(5) symmetry of the edge $l_3^{v_1}$ if $\varphi(v_1) = \{l_1^{v_1} \prec l_2^{v_1} = l_1^{v_2} \vdash l_3^{v_1}\}$, $\varphi(v_2) = \{l_1^{v_2} \prec l_2^{v_2} \vdash l_3^{v_2}\}$, ie the isomorphism $\delta$, such that $\varphi(\delta(v_1)) = \{l_1^{\delta(v_1)} \prec l_2^{\delta(v_1)} \vdash l_3^{\delta(v_1)} = l_1^{\delta(v_2)}\}$, $\varphi(\delta(v_2)) = \{l_1^{\delta(v_2)} \prec l_2^{\delta(v_2)} \vdash l_3^{\delta(v_2)}\}$;
(6) when $\varphi(v_1) = \{e_1^{v_1} \prec e_2^{v_1} = e_1^{v_2} \vdash e_3^{v_1}\}$, $\varphi(v_2) = \{e_1^{v_2} \prec e_2^{v_2} \vdash e_3^{v_2}\}$, the raising the vertex $v_1$ on the edge $e_3^{v_2}$, ie the isomorphism $\eta$, such that $\eta(v_2) \prec \eta(v_1)$ and $\varphi(\eta(v_2)) = \{e_1^{\eta(v_2)} \prec e_2^{\eta(v_2)} \vdash e_3^{\eta(v_2)} = e_1^{\eta(v_1)}\}$, $\varphi(\eta(v_1)) = \{e_1^{\eta(v_1)} \prec e_2^{\eta(v_1)} \vdash e_3^{\eta(v_1)}\}$;
(7) when $\varphi(v_1) = \{e_1^{v_1} \prec l_1^{v_1} = l_1^{v_2}\}$, $\varphi(v_2) = \{l_1^{v_2} \prec l_2^{v_2} \vdash l_3^{v_2}\}$, the isomorphism $\mu$, such that $\mu(v_2) \prec \mu(v_1)$ and $\varphi(\mu(v_2)) = \{e_1^{\mu(v_2)} \prec l_1^{\mu(v_2)} \vdash e_2^{\mu(v_2)} = e_1^{\mu(v_1)}\}$, $\varphi(\mu(v_1)) = \{e_1^{\mu(v_1)} \prec l_1^{\mu(v_1)}\}$;
(8) when $\varphi(v_1) = \{l_1^{v_1} \prec l_2^{v_1} = l_1^{v_2} \vdash l_3^{v_1}\}$, $\varphi(v_2) = \{l_1^{v_2} \prec e_1^{v_2}\}$, the isomorphism $\nu$, such that $\nu(v_2) \prec \nu(v_1)$ and $\varphi(\nu(v_2)) = \{l_1^{\nu(v_2)} \prec e_1^{\nu(v_2)} = e_1^{\nu(v_1)}\}$, $\varphi(\nu(v_1)) = \{e_1^{\nu(v_1)} \prec l_1^{\nu(v_1)} \vdash e_2^{\nu(v_1)}\}$.

**Theorem 7.** *There exists the simple deformation of codimension 1 between the functions $f, g \in \Omega(M)$ if and only if exists the simple deformation between their Kronrod–Reeb graph $\Gamma_f$ and $\Gamma_g$.*

## Rerefences

[1] B.I.Hladysh, A.O.Prishlyak. Functions with nondegenerated critical points on the boundary of the surface. *Ukrainian Mathematical Journal*, 68(1) : 28–37, 2016.

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