The differential-geometric and algebraic aspects of the Lax-Sato theory

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In the report we investigate the Lax-Sato compatible systems of vector field equations, the related Lie algebraic structures and integrability properties of a very interesting class of nonlinear dynamical systems called the heavenly equations, which were initiated by Plebański and later analyzed in a series of articles [1, 2, 3]. Based on the Adler-Kostant-Symes-algebraic and related $\mathcal{R}$-structure schemes [4], applied to the holomorphic loop Lie algebra $\tilde{\mathcal{G}} := \widetilde{diff}(\mathbb{T}^n)$ of vector fields on torus $\mathbb{T}^n$, $n \in \mathbb{Z}_+$, there are studied orbits of the corresponding coadjoint actions on $\tilde{\mathcal{G}}^*$, deeply related with the classical Lie-Poisson type structures on them. Constructing two commuting to each other flows on the coadjoint space $\tilde{\mathcal{G}}^*$, generated by a chosen seed element $\tilde{l} \in \tilde{\mathcal{G}}^*$ and some Casimir invariants, we demonstrate successively that their compatibility condition coincides exactly with the corresponding heavenly equation under consideration.

As a by-product of the construction, devised in the paper [5], we state that all the heavenly equations allow such an origin and can be equivalently represented as a Lax type compatibility condition for specially built loop vector fields on the torus $\mathbb{T}^n$. We analyze the structure of the infinite hierarchy of conservations laws, related with the heavenly equations, and demonstrate their analytical structure connected with the Casimir invariants, generated by the Lie-Poisson structure on $\tilde{\mathcal{G}}^*$. Moreover, we extend the initial Lie-algebraic structure on the case when the basic Lie algebra $\mathcal{G} := diff(\mathbb{T}^n)$ is replaced by the adjacent holomorphic Lie algebra $\bar{\mathcal{G}} := diff_{hol}(\mathbb{C} \times \mathbb{T}^n) \subset diff(\mathbb{C} \times \mathbb{T}^n)$ of vector fields on $\mathbb{C} \times \mathbb{T}^n$. For all cases there are presented typical examples of the heavenly equations and demonstrated in details their integrability within the scheme devised in the paper. The latter also makes it possible to derive from the natural point of view the well known Lax-Sato representation for an infinite hierarchy of the heavenly type equations, related with the canonical Lie-Poisson structure on the adjoint space $\bar{\mathcal{G}}^*$. We discuss briefly the Lagrangian representation of these equations, following from their Hamiltonicity with respect to both deeply related commuting to each other evolution flows, the related bi-Hamiltonian structure as well as the Backlund transformations. It should be noted that there are only few examples of multi-dimensional integrable systems, whose mathematical structure was explained and in detail.

The heavenly type equations make up an important class of such integrable systems since some of them are obtained by a reduction of the Einstein equations with Euclidean (and neutral) signature for (anti-) self-dual gravity which includes the theory of gravitational instantons (see, for example, [6]). This and other cases of important applications of multi-dimensional integrable equations strongly motivated us to study a such class of equations and the related mathematical structures. As a very interesting aspect of our approach to describing integrability of the heavenly type dynamical systems is its Lagrange-d'Alambert type mechanical interpretation.

The main motivating idea which featured our report was based both on the article by P. Kulish [7], devoted to studying the super-conformal Korteweg-de-Vries equation as an integrable Hamiltonian flow on the adjoint space to the holomorphic loop Lie algebra of super-conformal vector fields on the circle $\mathbb{S}^1$, and the article by V. Mikhalev [8], devoted to studying Hamiltonian structures on the adjoint space to the holomorphic loop Lie algebra of smooth vector fields on the circle $\mathbb{S}^1$. In this connection the holomorphic loop Lie algebra of super-conformal vector fields on the circle $\mathbb{S}^{1|N}$, $N \in \mathbb{N}$, is used for constructing the integrable superanalogs of integrable heavenly type dynamical systems in the report.

Additionally we were strongly influenced both by the paper of M. Pavlov and L. Bogdanov [2] and by the paper of E. Ferapontov and J. Moss [9], in which they devised new effective differential-geometric and analytical methods for studying an integrable degenerate multi-dimensional dispersionless heavenly type equations, and whose mathematical meaning is still far from being properly appreciated.

Concerning the papers developing other Lie-algebraic approaches to constructing integrable heavenly type equations we mention papers by V. Ovsienko and C. Roger [10], B. Szablikowski and A. Sergyeyev [11] and by B. Kruglikov and O. Morozov [12].

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