Artin-Schreier coverings, Galois representations and density Sato-Tate distribution functions

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Let k be a field of characteristic $p > 0$ and let X be a nonsingular projective variety defined over k. In most cases $k = \mathbb{F}_q, q = p^s$ for some positive integer s.

An Artin-Schreier covering of X is a finite morphism $\pi : Y \to X$ from a normal variety Y onto X such that the field extension $k(Y)/k(X)$ is an Artin-Schreier extension. This extension is defined by the Artin-Schreier equation $y^p - y = f$ and $f \in k(X)$.

Example 1. The equation in affine form $y^p - y = cx + \frac{d}{x},\ c, d \in \mathbb{F}_p^*$ defines an Artin-Schreier covering of the projective line.

Example 2. The equation in affine form $y^p - y = x^n + a_1 x^{n-1} + \cdots + a_{n-1}x + a_n,\ a_i \in \mathbb{F}_p$ defines an Artin-Schreier curve and the respective covering of the projective line.

Proposition 3. *See* [1, 5]. *Let X be reduced absolutely irreducible nonsingular projective algebraic curve of genus g defined over a finite field $\mathbb{F}_q$. Let $\overline{\mathbb{F}}_q$ be an algebraic closure of $\mathbb{F}_q$. We may regard X as an algebraic curve over $\overline{\mathbb{F}}_q$. In this case on can attach to X the Jacobean $J(X)$ of X. Formal completion of the variety defines the commutative formal group F of dimension g.*

Proposition 4. *Under the conditions of the Proposition 3 there is an algebraic construction of the continuation of the Artin-Schreier covering $\pi : Y \to X$ to the map $\pi^{`} : J(Y) \to J(X)$.*

Let l be a prime such that $l \neq p$. Let $T_l(A)$ be the Tate module of the abelian variety A. In the case of the Jacobean $J(X)$ the Tate module is a free module of the rank $2g$ over l-adic numbers.

Example 5. For any two abelian varieties A_1 and A_2 the canonical homomorphism

$$h_l : Hom(A_1, A_2) \to Hom_{\mathbb{Z}_l}(T_l(A_1), T_l(A_2))$$

is an l-adic representation.

As the formal completion of an abelian variety defines a commutative formal group law (formal group) [2] F, present here some our results about homomorphisms of the groups over local and finite fields.

Let $\mathcal{O}$ be the complete discrete valuation ring of characteristic 0 with maximal ideal $\mathcal{M} = \pi\mathcal{O}$ so the residue field $k = \mathcal{O}/\mathcal{M}$ has characteristic $p > 0$. Let dimensions of formal groups F and G over $\mathcal{O}$ equal respectively n and m. Let $M_{nm}(\mathcal{O})$ be the ring of $n \times m$ matrices over $\mathcal{O}$.

Proposition 6. *Lubin homomorphism $c : Hom_{\mathcal{O}}(F, G) \to M_{nm}(\mathcal{O})$ is the injection.*

Let $Iso_R(F, G)$ be the set of isogenies from F to G over a commutative ring R.

Lemma 7. *If heights of group laws F and G are different then the set $Iso_k(F, G)$ is empty.*

Let F^* be the reduction of F by $mod\ \mathcal{M}$.

Proposition 8. *The mapping $* : Iso_{\mathcal{O}}(F, G) \to Iso_k(F^*, G^*)$ is the injection if $[p]^*$ is the isogeny.*

Definition 9. Let

$$T_p(c,d) = \sum_{x=1}^{p-1} e^{2\pi i(\frac{cx+\frac{d}{x}}{p})},$$
$$1 \le c, d \le p-1; \qquad x, c, d \in \mathbf{F}_p^*$$

be a Kloosterman sum.

Proposition 10. *The Kloosterman sum $T_p(c,d)$ is defined by the Artin-Schreier covering*

$$y^p - y = cx + \frac{d}{x}, \qquad c, d \in \mathbb{F}_p^*.$$

By A. Weil estimate $T_p(c,d) = 2\sqrt{p}\cos\theta_p(c,d)$.

There are possible two distributions of angles $\theta_p(c,d)$ on semiinterval $[0,\pi)$:

a) *p is fixed and c and d varies over $\mathbf{F}_p^*$; what is the distribution of angles $\theta_p(c,d)$ as $p \to \infty$;*

b) *c and d are fixed and p varies over all primes not dividing c and d.*

For the case *a*) N. Katz [3] and A. Adolphson [4] for few others sums proved that θ are distributed on $[0,\pi)$ with the Sato-Tate density $\frac{2}{\pi}\sin^2 t$.

Conjecture 11. *In the case b) when p varies over all primes then angles $\theta_p(1,1)$ are distributed on the interval $[0,\pi)$ with the Sato-Tate density $\frac{2}{\pi}\sin^2 t$.*

Rerefences

[1] David Mumford. Abelian Varieties. *Tata Institute of Fundamental Research Publications*, 2012, vol. 13, 263 p.
[2] Michiel Hazewinkel. *Formal Groups and Applications*, Providence, Rhode Island, Academic Press, 2012.
[3] Nicholas Katz. *Gauss sums, Kloosterman sums, and monodromy groups.* Princeton university press, 1988, 246 p.
[4] Alan Adolphson. *On the distribution of angles of Kloosterman sums.* Journ. rein. angew. Math., 1989, vol. 395, P. 214–220.
[5] Nikolaj Glazunov. *Extremal forms and rigidity in arithmetic geometry and in dynamics*, Chebyshevskii Sbornik, 2015, vol.16, no. 3, P. 124–146.