Twistors, harmonic spinors and symmetry operators

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In an n-dimensional spin manifold M, there are two first-order differential operators that can be defined on spinor fields which are written in terms of the Levi-Civita connection ∇. The first one is the Dirac operator

$$\not{D} = e^a.\nabla_{X_a}$$

where $\{X_a\}$ is the frame basis, $\{e^a\}$ is the coframe basis dual to it and . denotes the Clifford product. The second one is the twistor or Penrose operator defined as

$$P_X = \nabla_X - \frac{1}{n}\widetilde{X}.\not{D}$$

with respect ot any vector field X and its metric dual $\widetilde{X}$.

The spinor fields which are in the kernels of the Dirac operator and twistor operator are called harmonic spinors and twistor spinors, respectively. They satisfy the following massless Dirac and twistor equations

$$\begin{aligned} \not{D}\psi &= 0 \\ \nabla_X\psi &= \frac{1}{n}\widetilde{X}.\not{D}\psi \end{aligned}$$

respectively, where ψ is a spinor field.

Symmetry operators are defined as the operators that take solutions of an equation and give another solution and they can be constructed for harmonic spinors and twistor spinors from conformal Killing-Yano (CKY) forms which are antisymmetric generalizations of conformal Killing vector fields to higher degree differential forms [1, 2]. A p-form ω is a CKY p-form if it satisfies the equation

$$\nabla_X\omega = \frac{1}{p+1}i_X d\omega - \frac{1}{n-p+1}\widetilde{X}\wedge\delta\omega$$

where i_X is the contraction operator with respect to the vector field X, d is the exterior derivative and δ is the co-derivative operators. The symmetry operators of harmonic spinors are written in terms of a CKY p-form as

$$\mathcal{L}_\omega = i_{X^a}\omega.\nabla_{X_a} + \frac{p}{2(p+1)}d\omega - \frac{n-p}{2(n-p+1)}\delta\omega$$

and the symmetry operators of twistor spinors are

$$L_\omega = -(-1)^p\frac{p}{n}\omega.\not{D} + \frac{p}{2(p+1)}d\omega + \frac{p}{2(n-p+1)}\delta\omega.$$

Transformation operators that transform twistor spinors to harmonic spinors are also constructed as

$$L_\alpha = \alpha.\not{D} + \frac{(-1)^p n}{n-2(p+1)}d\alpha - \frac{(-1)^p n}{n-2(p-1)}\delta\alpha$$

where in this case α is a potential p-form that satisfies a generalized equation which reduces to Laplace equation for $p=0$.

These constructions are generalized to $Spin^c$ manifolds in which gauged harmonic spinors and gauged twistor spinors are defined in terms of the gauged covariant derivative $\widehat{\nabla}_X = \nabla_X + i_X A$ and the gauged Dirac operator $\widehat{\not{D}} = \not{D} + A$ where A is the gauge connection 1-form. Algebraic conditions to obtain solutions of the Seiberg-Witten equations from gauged harmonic spinors are determined [3, 4].

Rerefences

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