Finiteness of pretangent spaces at infinity

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Let (X,d) be an unbounded metric space, p be a point of X and $\tilde{r}=(r_n)_{n\in\mathbb{N}}$ be a scaling sequence of positive real numbers tending to infinity. Denote by $\tilde{\boldsymbol{X}}_{\infty,\tilde{r}}$ the set of all sequences $\tilde{x}=(x_n)_{n\in\mathbb{N}}\subset X$ for each of which $\lim\limits_{n\to\infty} d(x_n,p)=\infty$ and there is a limit $\tilde{\tilde{d}}_{\tilde{r}}(\tilde{x})=\lim\limits_{n\to\infty}\frac{d(x_n,p)}{r_n}$.

Define the equivalence relation $\equiv$ on $\tilde{\boldsymbol{X}}_{\infty,\tilde{r}}$ as

$$(\tilde{x}\equiv\tilde{y}) \Leftrightarrow \left(\lim_{n\to\infty}\frac{d(x_n,y_n)}{r_n}=0\right).$$

Let $\boldsymbol{\Omega}^X_{\infty,\tilde{r}}$ be the set of equivalence classes generated by $\equiv$ on $\tilde{\boldsymbol{X}}_{\infty,\tilde{r}}$. We shall say that points $\alpha,\beta\in\boldsymbol{\Omega}^X_{\infty,\tilde{r}}$ are *mutually stable* if for $\tilde{x}\in\alpha$ and $\tilde{y}\in\beta$ there is a finite limit

$$\rho(\alpha,\beta):=\lim_{n\to\infty}\frac{d(x_n,y_n)}{r_n}. \tag{1}$$

Let us consider the weighted graph $(G_{X,\tilde{r}},\rho)$ with the set of vertices $V(G_{X,\tilde{r}})=\boldsymbol{\Omega}^X_{\infty,\tilde{r}}$, the edge set $E(G_{X,\tilde{r}})$ such that

$$(\{u,v\}\in E(G_{X,\tilde{r}}))\Leftrightarrow (u\,\text{and}\,v\,\text{are mutually stable and}\,u\neq v),$$

and the weight $\rho: E(G_{X,\tilde{r}})\to\mathbb{R}^+$ defined by formula (1).

Definition 1. The *pretangent spaces* (to (X,d) at infinity w.r.t. $\tilde{r}$) are the maximal cliques of $(G_{X,\tilde{r}},\rho)$ with metrics defined by (1).

Recall that a *clique* in a graph G is a set $A\subseteq V(G)$ such that every two distinct points of A are adjacent. A *maximal clique* in G is a clique C such that the inclusion

$$V(C)\subseteq V(A)$$

implies the equality $V(C)=V(A)$ for every clique A in G.

We study the conditions under which pretangent spaces are finite.

For natural $n\geq 2$ define the function $F_n: X^n\to\mathbb{R}$ by the rule

$$F_n(x_1,...,x_n):=\begin{cases}\dfrac{\min\limits_{1\leq k\leq n} d(x_k,p)\prod\limits_{1\leq k<l\leq n} d(x_k,x_l)}{\left(\max\limits_{1\leq k\leq n} d(x_k,p)\right)^{\frac{n(n-1)}{2}+1}} & \text{if } (x_1,...,x_n)\neq(p,...,p),\\ 0 & \text{if } (x_1,...,x_n)=(p,...,p).\end{cases}$$

Theorem 2. *Let (X,d) be an unbounded metric space, $p\in X$ and let $n\geq 2$. Then the inequality $\left|\Omega^X_{\infty,\tilde{r}}\right|\leq n$ holds for every pretangent space $\Omega^X_{\infty,\tilde{r}}\subseteq\boldsymbol{\Omega}^X_{\infty,\tilde{r}}$ if and only if*

$$\lim_{x_1,\dots,x_n\to\infty} F_n(x_1,...,x_n)=0.$$

Note that for every unbounded metric space (X,d) there is a pretangent space $\Omega^X_{\infty,\tilde r}$ consisting at least two points.

Let $(n_k)_{k\in\mathbb{N}}\subset\mathbb{N}$ be an infinite and strictly increasing sequence. Denote by $\tilde r'$ the subsequence $(r_{n_k})_{k\in\mathbb{N}}$ of the scaling sequence $\tilde r=(r_n)_{n\in\mathbb{N}}$ and, for every $\tilde x=(x_n)_{n\in\mathbb{N}}\in\tilde{\boldsymbol{X}}_{\infty,\tilde r}$, write $\tilde x':=(x_{n_k})_{k\in\mathbb{N}}$. It is clear that $\tilde{\tilde d}_{\tilde r'}(\tilde x')=\tilde{\tilde d}_{\tilde r}(\tilde x)$ for every $\tilde x\in\tilde{\boldsymbol{X}}_{\infty,\tilde r}$. Moreover, if $\tilde y\in\tilde{\boldsymbol{X}}_{\infty,\tilde r}$ and there is $\lim\limits_{n\to\infty}\frac{d(x_n,y_n)}{r_n}$, then

$$\lim_{k\to\infty}\frac{d(x_{n_k},y_{n_k})}{r_{n_k}}=\lim_{n\to\infty}\frac{d(x_n,y_n)}{r_n}.$$

Consequently, the map $\pi':\boldsymbol{\Omega}^X_{\infty,\tilde r}\to\boldsymbol{\Omega}^X_{\infty,\tilde r'}$ with $\pi'(\alpha)=\{(x_{n_k})_{k\in\mathbb{N}}:(x_n)_{n\in\mathbb{N}}\in\alpha\}$ is an embedding of $(G_{X,\tilde r},\rho)$ in $(G_{X,\tilde r'},\rho)$, i.e. if v and u are adjacent in $G_{X,\tilde r}$, then $\pi'(v)$ and $\pi'(u)$ are adjacent in $G_{X,\tilde r'}$ and $\rho(\{u,v\})=\rho(\{\pi'(u),\pi'(v)\})$. Hence, $\pi'(C)$ is a clique in $G_{X,\tilde r'}$ if C is a clique in $G_{X,\tilde r}$.

Definition 3. A pretangent space $\Omega^X_{\infty,\tilde r}$ is *tangent* if $\pi'(\Omega^X_{\infty,\tilde r})$ is a maximal clique in $G_{X,\tilde r'}$ for every infinite, strictly increasing sequence $(n_k)_{k\in\mathbb{N}}\subset\mathbb{N}$.

Theorem 2 gives us a condition under which all pretangent spaces $\Omega^X_{\infty,\tilde r}$ are finite. Let us consider now the problem of existence of finite tangent space.

Definition 4. Let $E\subseteq\mathbb{R}^+$. *The porosity* of E at infinity is the quantity

$$p(E,\infty):=\limsup_{h\to\infty}\frac{l(\infty,h,E)}{h}$$

where $l(\infty,h,E)$ is the length of the longest interval in the set $[0,h]\setminus E$. The set E is *strongly porous* at infinity if $p(E,\infty)=1$.

The standard definition of the porosity at a point can be found in [3].

Let (X,d) be a metric space and let $p\in X$. Write $S_p(X):=\{d(x,p):x\in X\}$.

Theorem 5. *Let (X,d) be an unbounded metric space, $p\in X$. The following statements are equivalent:*

(i) *The set $S_p(X)$ is strongly porous at infinity;*
(ii) *There is a single-point, tangent space $\Omega^X_{\infty,\tilde r}$;*
(iii) *There is a finite tangent space $\Omega^X_{\infty,\tilde r}$;*
(iv) *There is a compact tangent space $\Omega^X_{\infty,\tilde r}$;*
(v) *There is a bounded, separable tangent space $\Omega^X_{\infty,\tilde r}$.*

Some results similar to Theorem 2 and Theorem 5 can be found in [1] and [2] respectively.

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Rerefences

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