Some applications of the discriminant and resonance sets of a real polynomial

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Let $f_n(x)$ be a monic polynomial of degree n with real coefficients $f_n(x) \equiv x^n + a_1 x^{n-1} + \cdots + a_n$. The n-dimensional space $\Pi \equiv \mathbb{R}^n$ of its coefficients $\mathbf{P} = (a_1, \dots a_n)$ is called the *coefficient space* of $f_n(x)$. A pair of roots t_i, t_j, $i, j = 1, \dots, n$, $i \neq j$, of $f_n(x)$ is called *$p:q$-commensurable* if $t_i : t_j = p : q$.

Definition 1. Resonance set $\mathcal{R}_{p:q}(f_n)$, $p \in \mathbb{Z}\backslash\{0\}$, $q \in \mathbb{N}$ of $f_n(x)$ is called the set of all points of the coefficient space Π at which $f_n(x)$ has at least a pair of $p:q$-commensurable roots, i.e.

$$\mathcal{R}_{p:q}(f_n) = \{\mathbf{P} \in \Pi : \exists i, j = 1, \dots, n, t_i : t_j = p : q\}. \tag{1}$$

The special case of $\mathcal{R}_{p:q}(f_n)$ at the $p = q = 1$ is so called *discriminant set* $\mathcal{D}(f_n)$, playing an important role in solution of many problems.

The polynomial $f_n(x)$ has a pair of $p:q$-commensurable roots iff the pair of polynomials $f_n(px)$ and $f_n(qx)$ has at least one common root, or in terms of resultant $\operatorname{Res}_x(f_n(px), f_n(qx)) = 0$. In the case when $p = q$ both polynomials $f_n(px)$ and $f_n(qx)$ have exactly n common roots. In case $a_n = 0$ one of the root is equal to zero, therefore resultant can be written in the form $\operatorname{Res}_x(f_n(px), f_n(qx)) = a_n(p-q)^n \mathrm{GD}_{p:q}(f_n)$, where $\mathrm{GD}_{p:q}(f_n)$ is so called *generalized discriminant* of the polynomial $f_n(x)$ introduced in [1].

Definition 2. The *chain* $\mathrm{Ch}^{(k)}_{p:q}(t_i)$ *of $p:q$-commensurable roots* of length k (shortly *chain of roots*) is called the finite part of geometric progression with common ratio p/q and scale factor t_i, each member of which is a root of polynomial $f_n(x)$. The value t_i is called the *generating root*.

The detail structure of the resonance set (1) can be described with the help of so called *i-th generalized subdiscriminants* $\mathrm{GD}^{(i)}_{p:q}(f_n)$, which are nontrivial factors of i-th subresultants of pair of polynomials $f_n(px)$ and $f_n(qx)$. Such subresultants can be computed as i-th inners of Sylvester matrix constructed from the coefficients of mentioned above polynomials. For more details see [2].

Theorem 3. *Polynomial $f_n(x)$ has exactly $n-d$ different chains of roots $\mathrm{Ch}^{(i)}_{p:q}(t_j)$, $j = 1, \dots, n-d$ iff in the sequence $\left\{\mathrm{GD}^{(i)}_{p:q}(f_n), i = 0, \dots, n-1\right\}$ of i-th generalized subdiscriminants of $f_n(x)$ the first nonzero subdiscriminant is d-th generalized subdiscriminant $\mathrm{GD}^{(d)}_{p:q}(f_n)$.*

Consider a partition $\lambda = [1^{n_1} 2^{n_2} \dots i^{n_i} \dots]$ of $n \in \mathbb{N}$. Partition functions $p(n)$ and $p_l(n)$ return the number of all partitions and the number of all partitions of the length l of $n \in \mathbb{N}$ respectively. The value i in the partition λ defines the length of chain $\mathrm{Ch}^{(i)}_{p:q}(t_i)$ for a corresponding generating root t_i, the value n_i defines the number of different generating roots, which give the chains of root of the length i. Any partition λ of number n defines a certain structure of $p:q$-commensurable roots of this polynomial and it corresponds to some algebraic variety $\mathcal{V}^i_l$, $i = 1, \dots, p_l(n)$ of dimension l in the coefficient space Π. The number of such varieties of dimension l is equal to $p_l(n)$ and total number of all varieties consisting the resonance set $\mathcal{R}_{p:q}(f_n)$ is equal to $p(n) - 1$.

Parametrization of variety $\mathcal{V}_1$ can be expressed in q-binomial (Gaussian) coefficients:

$$a_i = (-1)^i \begin{bmatrix} n \\ i \end{bmatrix}_q p^{\frac{1}{2}i(i-1)} q^{\frac{1}{2}i(2n-i(i+1))} t_1^i, \quad i = 1, \dots, n, \text{ where } \begin{bmatrix} n \\ k \end{bmatrix}_q = \prod_{i=1}^{k} \frac{q^{n-i+1} - 1}{q^i - 1}.$$

Computation of parametric representation of any variety $\mathcal{V}_l$, $2 \le l \le n-1$, from the resonance set $\mathcal{R}_{p:q}(f_n)$ is based on the following

Theorem 4 ([3]). *Let $\mathcal{V}_l$, $\dim \mathcal{V}_l = l < n-1$, be a variety on which polynomial $f_n(x)$ has l different chains of $p:q$-commensurable roots and the chain generated by the root t_1 has length $m>1$. Let denote by $\mathbf{r}_l(t_1, t_2, \dots, t_l)$ parametrization of variety $\mathcal{V}_l$. Therefore the following formula*

$$\mathbf{r}_{l+1}(t_1, \dots, t_l, t_{l+1}) = \mathbf{r}_l(t_1, \dots, t_l) + \frac{p\left(t_{l+1} - p^{m-1} t_1\right)}{t_1\left(p^m - q^m\right)}\left[\mathbf{r}_l(t_1, \dots, t_l) - \mathbf{r}_l((q/p)t_1, \dots, t_l)\right] \qquad (2)$$

gives parametrization of the part of variety $\mathcal{V}_{l+1}$, on which there exists $\mathrm{Ch}_{p:q}^{(m-1)}(t_1)$, simple root t_{l+1} and other chains of roots are the same as on the initial variety $\mathcal{V}_l$.

From the geometrical point of view Theorem 4 means that a part of variety $\mathcal{V}_{l+1}$ is formed as a ruled $l+1$-dimensional surface by the secant lines, which cross its directrix $\mathcal{V}_l$ at two points defined by such values of parameters t_1^1 and t_1^2 that $t_1^1 : t_1^2 = q : p$. At $p/q \to 1$ mentioned above ruled surface becomes a tangent ruled surface which parametrization is $\mathbf{r}_{l+1} = \mathbf{r}_l + m^{-1}(t_{l+1} - t_1)\partial \mathbf{r}_l / \partial t_1$. If $f_n(x)$ has on the variety $\mathcal{V}_{l+1}$ a pair of complex-conjugate roots it is necessary to make continuation of obtained parametrization (2). Finally, it is possible to pass from variety with two chains of roots of length k to a variety with a chain of roots of length $2k$. Thus, combining the mentioned above procedure one can state the following

Theorem 5 ([3]). *Resonance set $\mathcal{R}_{p:q}(f_n)$ of a real polynomial $f_n(x)$ for a certain value of commensurability coefficient $p:q$ allows polynomial parametrization of each variety $\mathcal{V}_l \subset \mathcal{R}_{p:q}(f_n)$, $l = 1, \dots, n-1$.*

The software library for computation of the resonance set $\mathcal{R}_{p:q}(f_n)$ was implemented for CAS `Maple`. The above results were effectively used in solving the following problems.

(1) The resonance set $\mathcal{R}_{p:q}(f_3)$ of a cubic was completely described and an outline of investigation of formal stability of a stationary point of a multiparameter Hamiltonian system with three degrees of freedom was proposed [4].
(2) The discriminant set of a real cubic polynomial was used in computation of global parametrization of one real variety Ω that plays an important role in the investigation of the normalized Ricci flow on generalized Wallach spaces related to invariant Einstein metrics [5].
(3) Parametric representation of the discriminant set $\mathcal{D}(f_4)$ of a quartic allows to find the set of stability of the linearized multiparameter Hamiltonian system with 4 degrees of freedom [6].

This talk is devoted to the description of the resonance $\mathcal{R}_{p:q}(f_4)$ and discriminant $\mathcal{D}(f_4)$ sets of a real quartic $f_4(x)$ and their application to the problem (3) in nonlinear case.

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