# Construction and topological properties of the closed extension topology

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Let $(X,\tau)$ be a topological space and let $X^*$ be a superset of $X$. Then the family

$$\tau^* = \{V \subset X^* \mid V \subset X^* \backslash X \text{ or } V = (X^*\backslash X) \cup U,\ U \in \tau\}$$

is a topology for $X^*$, which is called *the closed extension topology of $X$ to $X^*$*. Indeed, if all of the sets $U_\alpha \in \tau^*$, $\alpha \in T$, lies in $X^*\backslash X$, then their union is also contained in $X^*\backslash X$. Otherwise, this union has the form $(X^*\backslash X) \cup U$ for some $U \in \tau$. If the index set $T$ is finite and at least one of $U_\alpha$ lies in $X^*\backslash X$, then the intersection $\bigcap\limits_{\alpha\in T} U_\alpha$ is also contained in $X^*\backslash X$. Otherwise, this intersection has the form $(X^*\backslash X) \cup U$ for some $U \in \tau$.

Thus the open sets of $X^*$ are all subsets of $X^*\backslash X$ and all unions $(X^*\backslash X) \cup U$ where $U$ is open in $X$. Respectively, closed sets in $X^*$ are all supersets of $X$ and all closed sets in $X$.

**Example 1.** Let $X$ be a topological space and let $A$ be a proper subset of $X$. Then $A$-excluded topology for $X$ is the closed extension topology of the indiscrete space $A$ to $X$.

**Proposition 2.** *The closed extension topology $\tau^*$ of a topological space $(X,\tau)$ to $X^*$ is supremum of topology $\sigma = \{\varnothing\} \cup \{(X^*\backslash X) \cup V,\ V \in \tau\}$ and $X$-excluded topology for $X^*$.*

A map $f : X \to Y$ of topological spaces $(X,\tau)$ and $(Y,\sigma)$ is called *inducing*, if the topology $\tau$ is induced by $\sigma$ and $f$.

**Theorem 3.** *Let $(X,\tau)$ be a topological space, let $X^*$ be a superset of the set $X$ and let $\tau^*$ be the closed extension topology of $X$ to $X^*$. Then the natural embedding $X \ni x \overset{i}{\mapsto} x \in X^*$ is closed inducing map. In particular, $(X,\tau)$ is closed subspace of $(X^*,\tau^*)$.*

Unlike extension topology [2] closed extension topology (like open extension topology [3]) is not transitive. Hence the natural embedding $X \ni x \overset{i}{\mapsto} x \in X^*$ is not quotient, i. e. the closed extension topology is not quotient topology with respect to $\tau$ and $i$.

**Example 4.** Consider nested sets $X = \{a\}$, $X^* = \{a,b\}$, $X^{**} = \{a,b,c\}$. Then for the discrete topology $\tau$ for $X$ we have $(\tau^*)^* = \{\varnothing, X^{**}, \{c\}, \{b,c\}\} \neq \tau^{**} = \{\varnothing, X^{**}, \{b\}, \{c\}, \{b,c\}\}$.

Let us describe a base of the least cardinality of the closed extension topology and local one with respect to it.

**Proposition 5.** *A base of the least cardinality of the closed extension topology of a space $X$ to $X^*$ has the form $\beta^* = \{\{x\}, (X^*\backslash X) \cup U \mid x \in X^*\backslash X, U \in \beta\}$ where $\beta$ is a base of the least cardinality of the space $X$.*

**Proposition 6.** *Let $\tau^*$ be the closed extension topology of a space $X$ to $X^*$. Then a local base of the least cardinality at any point $x \in X^*$ has the form $\{\{x\}\}$ for $x \in X^*\backslash X$, and $\{(X^*\backslash X) \cup U \mid U \in \beta_x\}$ where $\beta_x$ is a local base of the least cardinality at point $x$ in $X$, for $x \in X$.*

The following Proposition gives an explicit description of the interior, the closure, the sets of isolated and limit points of an arbitrary set of a topological space with the closed extension topology, and necessary and sufficient conditions for density and nowhere density of a given set.

**Proposition 7.** *Let $\tau^*$ be the closed extension topology of a space $X$ to $X^*$ and let $A \subset X^*$. Then:*

*1) the interior of $A$ in $X^*$ is equal to $\mathrm{Int}_{X^*} A = (X^* \backslash X) \cup \mathrm{Int}_X(A \cap X)$, where $\mathrm{Int}_X(A \cap X)$ is the interior of the intersection $A \cap X$ in $X$, when $A \supset X^* \backslash X$, and $\mathrm{Int}_{X^*} A = A \backslash X$ otherwise;*

*2) the closure of $A$ in $X^*$ has the form $\overline{A}_{X^*} = \overline{A}_X$, where $\overline{A}_X$ is the closure of $A$ in $X$, when $A \subset X$, and $\overline{A}_{X^*} = A \cup X$ otherwise;*

*3) the set of isolated points of $A$ in $X^*$ is calculated by formula $\mathrm{I}_{X^*}(A) = \mathrm{I}_X(A)$, where $\mathrm{I}_X(A)$ is the set of isolated points of $A$ in $X$, when $A \subset X$, and $\mathrm{I}_{X^*}(A) = A \backslash X$ otherwise;*

*4) the set of limit points of $A$ in $X^*$ is equal to $A'_X$, where $A'_X$ is the set of limit points of $A$ in $X$, when $A \subset X$, and $A'_{X^*} = X$ otherwise;*

*5) if $X^* \backslash X \neq \varnothing$, then $A$ is dense in $X^*$ if and only if $A \supset X^* \backslash X$; otherwise $A$ is dense in $X^*$ if and only if $A$ is dense in $X$.*

*6) $A$ is nowhere dense in $X^*$ if and only if $A \subset X$.*

Now we proceed with the study of topological properties of the closed extension topology.

**Theorem 8.** *Let $\tau^*$ be the closed extension topology of a space $X$ to $X^*$ and $X^* \neq X$. Then $X^*$ is path connected and thus connected.*

**Corollary 9.** *Every topological space can be embedded as closed subspace in a path connected (and thus connected) topological space.*

The next fact follows directly from propositions 5 and 6.

**Theorem 10.** *Let $\tau^*$ be the closed extension topology of a space $X$ to $X^*$. The topological space $X^*$ is first countable if and only if the space $X$ is first countable. The space $X^*$ is second countable if and only if the space $X$ is second countable and the complement $X^* \backslash X$ is at most countable.*

**Theorem 11.** *Let $\tau^*$ be the closed extension topology of a space $X$ to $X^*$. The topological space $X^*$ is Lindelöf (compact) if and only if the space $X$ is Lindelöf (compact).*

**Theorem 12.** *Let $\tau^*$ be the closed extension topology of a space $X$ to $X^*$ and $X^* \neq X$. The topological space $X^*$ is separable if and only if the complement $X^* \backslash X$ is at most countable. The set $X^* \backslash X$ is at most countable dense set in $X^*$ of the least cardinality.*

**Theorem 13.** *Let $\tau^*$ be the closed extension topology of a space $X$ to $X^*$ and $X^* \neq X$. The topological space $X^*$ is $T_0$ if and only if the space $X$ is $T_0$. The space $X^*$ is $T_4$ if and only if every two nonempty closed sets in $X$ intersect (a stronger condition than $T_4$ separation axiom). The space $X^*$ is not $T_i$ for $i = 1, 2, 3$. In particular, $X^*$ is not regular, normal and metrizable.*

## Rerefences


[1] *Lynn Arthur Steen and J. Arthur Seebach, Jr.* Counterexamples in topology. – New York: Dover publications, 1978. – 256 p.

[2] *Babych V. M., Filonenko O. M.* Extension Topology // Scientific Bulletin of Uzhhorod University. Series of Mathematics and Informatics. – 2015. – **1 (26)**. – p. 7-11. [in Ukrainian]

[3] *Babych V. M., Pyekhtyeryev V. O.* Open Extension Topology // Proc. Intern. Geom. Center. – 2015. – Vol. 8. – No. 2. – p. 20-25. [in Ukrainian]

[4] *Engelking R.* General Topology. Revised and completed edition. – Berlin: Heldermann Verlag, 1989. – 540 p.