Killing vector fields and geometry of submersions

Annaev Nuriddin Uzoq o'g'li
(Architecture and Construction Institute,Tashkent,Uzbekistan)
E-mail: nuriddin.annayev.91@mail.ru

Let M be a smooth Riemannian manifold of dimension n with the Riemannian metric g, ∇ – the Levi-Civita connection, $\langle \cdot, \cdot \rangle$ - inner product defined by the Riemannian metric g.

We denote by $V(M)$ the set of all smooth vector fields defined on M, through a $[X, Y]$ Lie bracket of vector fields $X, Y \in V(M)$. The set $V(M)$ is a Lie algebra with Lie bracket.

Throughout the paper, the smoothness means smoothness of a class C^∞.

Definition 1. Differentiable mapping $\pi : M \to B$ of a maximal rank, where B is smooth manifold of dimension m, $(n > m)$, is called submersion.

By the theorem on the rank of a differentiable function for each point $p \in B$ the full inverse image $\pi^{-1}(p)$ is a submanifold of M dimension $k = n - m$. Thus submersion $\pi : M \to B$ generates a foliation F on M of dimension $k = n - m$, whose leaves are connected components of submanifolds $L_p = \pi^{-1}(p), p \in B$.

To the study of the geometry of submersions were devoted numerous papers ([1]–[5]), in particular in paper [3] derived the fundamental equations of submersion.

Let F be a foliation of dimension k, where $0 < k < n$. We denote by L_p leaf of foliation F, passing through a point $q \in M$, where $\pi(q) = p$, by T_qF tangent space of leaf L_p at the point $q \in L_p$, by $H(q)$ orthogonal complement of subspace T_qF. As result arise subbundle's $TF = \{T_qF\}$, $TH = \{H(q)\}$ of the tangent bundle TM and we have an orthogonal decomposition $TM = TF \oplus TH$. Thus every vector field X is decomposable as: $X = X^v + X^h$, where $X^v \in TF$, $X^h \in TH$. If $X^h = 0$ (respectively $X^v = 0$), then the field X is called as vertical (respectively horizontal) vector field.

The submersion $\pi : M \to B$ is said to be Riemannian if differential $d\pi$ preserves lengths of horizontal vectors. It is known that Riemannian submersions generate Riemannian foliation [5].

We remark that foliation F is called Riemannian if every geodesic, orthogonal at some point to leaves, remains orthogonal to leaves at all points.

The curve is called as horizontal if it's tangential vector is horizontal.

Let $\gamma : [a, b] \to B$ is smooth curve in B, and $\gamma(a) = p$. Horizontal curve $\tilde{\gamma} : [a, b] \to M$, $\tilde{\gamma}(a) \in \pi^{-1}(p)$ is called as horizontal lift of a curve $\gamma : [a, b] \to B$, if $\pi(\tilde{\gamma}(t)) = \gamma(t)$ for all $t \in [a, b]$.

The map $S : V(F) \times H(F) \to V(F)$, defined by the formula $S(X, U) = \nabla^v_X U$, is called second basic tensor, where $\nabla^v_X U$ is vertical component of vector field $\nabla_X U$, $V(F)$, $H(F)$ set of vertical and horizontal vector fields respectively.

At the fixed field of normal $U \in HF$, map $S(X, U)$ generates tensor field S_U of type (1,1):

$$S(X, U) = S_U X = \nabla^v_X U.$$

The tensor field S_U defines the bilinear form l_U :

$$l_U(X, Y) = \langle S_U X, Y \rangle.$$

The form $l_U(X, Y)$ is called second basic form with respect to a normal vector field U.

The tensor field S_U is linear map and consequently it is defined by the matrix $S(X, U) = AX$.

Horizontal vector field U is called basic if vector field $[Y, U]$ is also vertical for each vector field $Y \in V(F)$. Eigenvalues of matrix A is called the principal curvature of foliation F, when vector field U is basic. If the principal curvatures are locally constant along leaf, then foliation F is called isoparametric.

Let's consider some set $D \subset V(M)$, which contains finite or infinite number of smooth vector fields. For a point $x \in M$ through $t \to X^t(x)$ we will denote the integral curve of a vector field X passing through a point x at $t = 0$. Map $t \to X^t(x)$ is defined in some domain $I(x) \subset R$, which generally depends on field X and point x.

Recall that the vector field X on M is called the Killing vector field, if the group of local transformations $x \to X^t(x)$ consists of isometries [2]. Geometry of Killing vector fields is subject of numerous studies in connection its importance in geometry and other areas of mathematics [1], [2].

We will denote through $A(D)$ the smallest Lie subalgebra of algebra $K(M)$, containing the set D. Since the algebra $K(M)$ finite dimensional, there exist vector fields $X_1,\ X_2, ..., X_m$, that vectors $X_1(x),\ X_2(x),\ ..., X_m(x)$ forms basis for the subspace $A_x(D)$ for each $x \in M$, where

$$A_x(D) = \{X(x) : X \in A(D)\}.$$

Results of [2] allows constructing various submersions $\pi : R^m \to L(x_0)$ using the vector fields $X_1, X_2, ..., X_m$, by the formula

$$\pi(t_1, t_2, \ldots, t_m) = X_m^{t_m}(X_{m-1}^{t_{m-1}}(\ldots(X_1^{t_1}(x_0)\ldots))).$$

Let's consider the Killing vector fields

$$Y_1 = \frac{\partial}{\partial x_1},\quad Y_2 = \frac{\partial}{\partial x_2},\quad Y_3 = -x_3\frac{\partial}{\partial x_1} + x_1\frac{\partial}{\partial x_3},\quad Y_4 = \frac{\partial}{\partial x_4}$$

on R^4. It is easy to check that the basis of subalgebra $A(D)$ consists of following vector fields

$$X_1 = \frac{\partial}{\partial x_1},\quad X_2 = \frac{\partial}{\partial x_3},\quad X_3 = -x_3\frac{\partial}{\partial x_1} + x_1\frac{\partial}{\partial x_3},\quad X_4 = \frac{\partial}{\partial x_2},\quad X_5 = \frac{\partial}{\partial x_4}$$

and consequently the orbit $L(p)$ for each point $p \in R^4$ coincides with space R^4.

We will define following submersion $\pi : R^5 \to R^4$ with formula

$$\pi(t_1,\, t_2,\, t_3,\, t_4,\, t_5) = X_5^{t_5}(X_4^{t_4}(X_3^{t_3}(X_2^{t_2}(X_1^{t_1}(O))))),$$

where $O-$origin of coordinates in R^4.

Theorem 2. *There exists a Riemannian metric $\tilde{g}$ on R^4 that:*
1) Map $\pi : R^5 \to R^4$ is Riemannian submersion;
2) Submersion $\pi : R^5 \to R^4$ generates on R^5 is isoparametric foliation;
3) $(R^4,\, \tilde{g})$ is manifold of nonnegative curvature.

Rerefences

[1] Zoyidov A.N., Tursunov B.A. Geometry of submersions on manifolds of nonnegative curvature. Uzbek mathematical journal. 2015, № 2, pp.27-34.

[2] Narmanov A.Ya., Saitova S.S. On the Geometry of Orbits of Killing Vector Fields. Differential Equations, 2014, Vol. 50, No. 12, pp. 1–8. Original Russian Text published in Differentsial'nye Uravneniya, 2014, Vol. 50, No. 12, pp. 1582-1589.

[3] O'Neil B. The Fundamental equations of a submersions. Michigan Mathematical Journal, v.13, 1966, p. 459-469

[4] Hermann R. A sufficient condition that a mapping of Riemannian manifolds to be a fiber bundle. Proc. Amer. Math. Soc.11 (1960), 236-242.

[5] Reinhart B. L. Foliated manifolds with bundle-like metrics. Annals of Mathematics, Second Series, Vol. 69, No. 1,1959, pp. 119-132

[6] Sussmann. H. Orbits of family of vector fields and integrability of systems with singularities. Bull. Amer. Math. Soc., 1973, 79, p. 197-199.