Boundary behavior of ring Q-homeomorphisms on Finsler manifolds

Elena Afanas'eva
(Institute of Applied Mathematics and Mechanics, 1 Dobrovol'skogo St., Slavyansk 84100, Ukraine)
E-mail: `es.afanasjeva@yandex.ru`

By a *Finsler manifold* $(\mathbb{M}^n, \Phi)$, $n \geq 2$, we mean a smooth manifold of class C^∞ with defined Finsler structure $\Phi(x,\xi)$, where $\Phi(x,\xi) : T\mathbb{M}^n \to \mathbb{R}^+$ is a function satisfying the following conditions:

1) $\Phi \in C^\infty(T\mathbb{M}^n \setminus \{0\})$;
2) for all $a > 0$ hold $\Phi(x, a\xi) = a\Phi(x,\xi)$ and $\Phi(x,\xi) > 0$ for $\xi \neq 0$;
3) the $n \times n$ Hessian matrix $g_{ij}(x,\xi) = \frac{1}{2}\frac{\partial^2 \Phi^2(x,\xi)}{\partial \xi_i \partial \xi_j}$ is positive defined at every point of $T\mathbb{M}^n \setminus \{0\}$, cf. [1].

By the *geodesic distance* $d_\Phi(x,y)$ we mean the infimum of lengths of piecewise-smooth curves joining x and y in $(\mathbb{M}^n, \Phi)$, $n \geq 2$.

Later we consider the Finsler structure of a type $\widetilde{\Phi}(x,\xi) = \frac{1}{2}(\Phi(x,\xi) + \Phi(x,-\xi))$ thereby obtaining a Finsler manifold $(\mathbb{M}^n, \widetilde{\Phi})$ with symmetrized (reversible) function $\widetilde{\Phi}$. Clearly, if $\widetilde{\Phi}$ is reversible, then the induced distance function $d_{\widetilde{\Phi}}$ is reversible, i.e., $d_{\widetilde{\Phi}}(x,y) = d_{\widetilde{\Phi}}(y,x)$, for all pairs of points $x, y \in \mathbb{M}^n$, see [2]. It is also known that the reversible Finsler metric coincides with the Riemannian one, see, e.g., [3].

Definition 1. The *modulus* of the family Γ is defined by

$$M(\Gamma) = \inf \int\limits_D \rho^n(x)\, d\sigma_{\widetilde{\Phi}}(x),$$

where the infimum is taken over all nonnegative Borel functions ρ such that the condition

$$\int\limits_\gamma \rho \widetilde{\Phi}(x, dx) = \int\limits_\gamma \rho ds_{\widetilde{\Phi}} \geq 1$$

holds for any curve $\gamma \in \Gamma$. The functions ρ, satisfying this condition, are called *admissible* for Γ, cf. [1].

Definition 2. Let D and D' be domains on the Finsler manifolds $(\mathbb{M}^n, \widetilde{\Phi})$ and $(\mathbb{M}^n_*, \widetilde{\Phi}_*)$, $n \geq 2$, respectively, and let $Q : \mathbb{M}^n \to (0,\infty)$ be a measurable function. A homeomorphism $f : D \to D'$ is *ring Q-homeomorphism at a point* $x_0 \in \overline{D}$, if

$$M\left(\Delta(f(C), f(C_0); D')\right) \;\leq \int\limits_{A\cap D} Q(x) \cdot \eta^n\left(d_{\widetilde{\Phi}}(x, x_0)\right) d\sigma_{\widetilde{\Phi}}(x) \tag{1}$$

holds for any geodesic ring $A = A(x_0, \varepsilon, \varepsilon_0)$, $0 < \varepsilon < \varepsilon_0 < \infty$, any two continua (compact connected sets) $C \subset \overline{B(x_0,\, \varepsilon)} \cap D$ and $C_0 \subset D \setminus B(x_0, \varepsilon_0)$ and each Borel function $\eta : (\varepsilon, \varepsilon_0) \to [0,\infty]$, such that $\int\limits_{\varepsilon}^{\varepsilon_0} \eta(r)dr \geq 1$. We say that f is a *ring Q-homeomorphism* in D, if (1) holds for all points $x_0 \in \overline{D}$.

Definition 3. We say that the boundary D is *strongly accessible at a point* $x_0 \in \partial D$, if for any neighborhood U of x_0, there are a compactum $E \subset D$, a neighborhood $V \subset U$ of x_0 and a number $\delta > 0$, such that $M(\Delta(E, F; D)) \geq \delta$ for any continuum F in D, intersecting ∂U and ∂V.

Theorem 4. *Let D and D' be domains in $(\mathbb{M}^n, \widetilde{\Phi})$ and $(\mathbb{M}^n_*, \widetilde{\Phi}_*)$, $n \geq 2$, respectively. Assume that D is locally connected at a point $x_0 \in \partial D$, $\overline{D'}$ is compact and the boundary of D' is strongly accessible. If* *a measurable function* $Q : \mathbb{M}^n \to (0, \infty)$ *satisfies*

$$\int\limits_{D(x_0,\varepsilon,\varepsilon_0)} \frac{Q(x) d\sigma_{\widetilde{\Phi}}(x)}{d_{\widetilde{\Phi}}(x, x_0)^n} = o\left(\left[\log \frac{1}{\varepsilon}\right]^n\right) \quad as \quad \varepsilon \to 0, \tag{2}$$

where $D(x_0, \varepsilon, \varepsilon_0) = \{x \in D : \varepsilon < d_{\widetilde{\Phi}}(x, x_0) < \varepsilon_0\}$ *for* $\varepsilon_0 < d(x_0) = \sup\limits_{x \in D} d_{\widetilde{\Phi}}(x, x_0)$*, then any ring* Q*-homeomorphism* $f : D \to D'$ *can be continuously extended to* x_0 *on* $(\mathbb{M}^n_*, \widetilde{\Phi}_*)$.

Corollary 5. *The assertion of Theorem 4 is true if the singular integral*

$$\int \frac{Q(x) d\sigma_{\widetilde{\Phi}}(x)}{d_{\widetilde{\Phi}}(x, x_0)^n} \tag{3}$$

converges in a neighborhood of the point x_0 *in the sense of principal value.*

Rerefences

[1] Yu. V. Dymchenko. The relation between the capacity of a condenser and the module of a family of separated surfaces in Finsler spaces. (Russian) *Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI)*, Analiticheskaya Teoriya Chisel i Teoriya Funktsii, 418(28): 74–89, 2013.

[2] X. Cheng and Z. Shen. *Finsler geometry. An approach via Randers spaces.* Heidelberg: Science Press Beijing, Beijing; Springer, 2012.

[3] D. Bao, S. Chern and Z. Shen. *An Introduction to Riemann-Finsler Geometry.* Graduate Texts in Mathematics, 200. New York: Springer-Verlag, 2000.