Natural sloshing frequencies in rigid truncated conical tanks

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Abstract
Purpose – The main purpose of this paper is to develop two efficient and accurate numerical analytical methods for engineering computation of natural sloshing frequencies and modes in the case of truncated circular conical tanks.

Design/methodology/approach – The numerical-analytical methods are based on a Ritz Treftz variational scheme with two distinct analytical harmonic functional bases.

Findings – Comparative numerical analysis detects the limit of applicability of variational methods in terms of the semi-apex angle and the ratio between radii of the mean free surface and the circular bottom. The limits are caused by different analytical properties of the employed functional bases. However, parallel use of two or more bases makes it possible to give an accurate approximation of the lower natural frequencies for relevant tanks. For V-shaped tanks, dependencies of the lowest natural frequency versus the semi-apex angle and the liquid depth are described.

Practical implications – The methods provide the natural sloshing frequencies for V-shaped tanks that are valuable for designing elevated containers in seismic areas. Approximate natural modes can be used in derivations of nonlinear modal systems, which describe a resonant coupling with structural vibrations.

Originality/value – Although variational methods have been widely used for computing the natural sloshing frequencies, this paper presents their application for truncated conical tanks for the first time. An original point is the use of two distinct functional bases.

Keywords Numerical analysis, Frequencies, Liquid flow containers

Paper type Research paper

1. Introduction
The number of constructions carrying a large liquid mass is enormous. Coupling the structure and liquid requires a precise and efficient computation of the natural sloshing

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(eigen-) frequencies and modes. Being subject of spacecraft applications, the natural sloshing frequencies and modes for conical tanks were studied in 1950-1960’s years of the past century. Special attention was paid to an estimate of the resulting hydrodynamic force and moment.

The international standards concerning the megaliter elevated tanks (Eurocode 8, 1998) have stated a typical design for concrete tanks of conical and conical-bottom shapes, typical examples are demonstrated by Damatty and Sweedian (2006). Owing to seismic events, liquid motions in a water tank on the supporting tower cause severe hydrodynamical loads. In the modelling of these loads, equivalent mechanical systems (Damatty and Sweedian, 2006; Dutta et al., 2004; Shrimali and Jangid, 2003) can be used. These systems relate liquid dynamics to oscillations of a pendulum or a spring-mass system. The eigenfrequencies of the equivalent systems should coincide with the lower natural sloshing frequencies and, therefore, an accurate prediction of the sloshing frequencies and modes is needed (Damatty et al., 2000; Dutta and Laha, 2000; Tang, 1999). This can be done by various Computational Fluid Dynamics (CFD) methods or by using semi-empirical approximate formulae (Damatty et al., 2000; Gavrilyuk et al., 2005). Although the CFD-methods demand lots of CPU’s power, they are primarily employed in engineering practise to guarantee a substantial precision. When concentrating on linear and nonlinear sloshing in V-shaped pure conical tanks, Gavrilyuk et al. (2005) showed that an alternative might consist in a semianalytical solution method. The method keeps the accuracy of the CFD-methods, but remains CPU-efficient and simple in use. The constructed numerical-analytical solutions may facilitate development of nonlinear sloshing theories (Lukovsky and Bilyk, 1985; Lukovsky, 1990, 2004; Bauer and Eidel, 1988; Faltinsen et al., 2000, 2003; Gavrilyuk et al., 2005, 2006). These theories are of importance for studying a resonant coupled vibration of a tower and the contained liquid.

In order to find the natural sloshing frequencies and modes, a spectral boundary value problem (Lukovsky et al., 1984; Ibrahim, 2005) has to be solved. When the tank is V- and L-shaped, the boundary value problem has no analytical solutions. Isolated analytical solutions exist only for pure (non-truncated) conical V-tanks. Such an example has been specified by Levin (1963) for the two lowest natural modes and the semiaxial angle \( \theta_0 = 45^\circ \). These modes are characterised by the wave number \( m = 1 \) in angular direction. Dokuchaev and Lukovsky (1968) generalised this result. They showed that analogous analytical solutions exist as \( \theta_0 = \arctan(\sqrt{m}) \). Mikishev and Rabinovich (1968) and Feschenko et al. (1969) used these solutions for evaluation of their numerical algorithms. Furthermore, simple approximate analytical solutions for pure conical V-tanks can be obtained by replacing the planar waterplane by a spherical segment. In that case, the spectral boundary value problem admits separation of variables in the spherical coordinate system. This has been realised by Dokuchaev (1964) and Bauer (1982). A satisfactory agreement with experimental data by Mikishev and Dorozhkin (1961) and Bauer (1982) was reported as \( \theta_0 = 15^\circ \).

The spectral boundary value problem for the linear sloshing modes admits a variational formulation (Feschenko et al., 1969). This variational formulation facilitates the Ritz-Trefz numerical scheme, whose practical realisation requires special sets of analytical harmonic functional bases. Two of these harmonic bases are used in the present paper for engineering computations of the natural frequencies and modes of liquid sloshing in truncated conical tanks. The first basis is of polynomial type.
(harmonic polynomial solutions, HPS). The second one employs the Legendre functions of first kind; it may be adopted by the mentioned nonlinear multimodal sloshing theories. Extensive numerical experiments have been done to identify all the geometrical parameters (semi-apex angle, position of secant plane and liquid depth), for which the proposed functional bases guarantee a sufficient number of significant figures for the lower natural frequencies.

In Section 3, the main focus is on the HPS. For the V-shaped tanks, we show that 11-17 HPS provide 4-6 significant digits of the lower natural frequencies for the semi-apex angles, which are smaller than 75° and larger than 10°. For the V- and Λ-shaped tanks which are characterised by semi-apex angles larger than 75°, the same number of the HPS guarantees 3-4 significant digits. For the Λ-shaped tanks with semi-apex angles smaller than 75°, convergence to the natural frequencies depends on the ratio between the radii of the mean liquid plane and the circular bottom. The method is not very efficient (only about 2-3 significant digits can be obtained with 17-20 basic polynomials) when the semi-apex angle is smaller than 60° and the ratio between the specified radii is smaller than 1/2. The slow convergence for the Λ-shaped tanks can partially be attributed to a singular asymptotic behaviour of the natural modes at the contact line formed by the waterplane and conical walls.

Lukovsky (1990) has proposed a non-conformal mapping technique to develop the multimodal method for the nonlinear sloshing problem in a non-cylindrical tank. Lukovsky and Timokha (2002) and Gavriluk et al. (2005) have realised this technique for a non-truncated V-tank. The natural sloshing modes were then approximated by a special functional basis (SFB). In Section 4, we use the curvilinear coordinate system proposed by Gavriluk et al. (2005) and generalise their results on natural sloshing frequencies to the case of truncated V- and Λ-shaped conical tanks. For typical geometrical configurations of elevated water tanks (a V-shaped cone with the semi-apex angle between 30° and 60°), the method shows faster convergence behaviour than in the case of Section 3. Six significant digits of the lowest sloshing frequency can be computed by using only 6-10 basis functions.

A comparative analysis of the two methods is presented in Section 5. In Section 6, we discuss the dependence of the lowest natural sloshing frequency on the geometrical shape of truncated conical tanks. Bearing in mind that the relevant water tanks are of a V-shape, we present the lowest spectral parameter (with a accuracy of five significant digits) versus the semi-apex angle and the ratio between the radii of the mean water plane and the bottom.

2. Statement of the problem
2.1 Differential and variational formulations
As it is typically assumed in sloshing theory (Ibrahim, 2005), we consider an ideal incompressible liquid with irrotational flow that partly occupies an earth-fixed rigid conical tank with the semi apex angle \( \theta_0 \). The mean (hydrostatic) liquid shape coincides with the domain \( Q_0 \) as it is shown in Figure 1. The gravity acceleration is directed downwards along the symmetry axis \( O_x \). The wetted conical walls are denoted by \( S_1 \). The circle \( S_2 \) is the tank bottom, \( S = S_1 \cup S_2 \) and \( \Sigma_0 \) is the non-perturbed (hydrostatic) waterplane. The origin is superposed with artificial apex of the conical surface. Henceforth, the problem is considered in a sizeless statement assuming that \( r_0 \) (the bottom radius for Λ-shaped and the water plane radius for V-shaped tanks)
is chosen as a characteristic geometrical dimension. Scaling by \( r_0 \) implies \( h = h/r_0 \rightarrow 0 \) (\( h \) is the liquid depth) and \( r_1 = r_1/r_0 \). The non-dimensional radius \( r_1 \) and the angle \( \theta_0 \) completely determine the geometric proportions of \( Q_0 \). The limit \( r_1 \rightarrow 1 \) implies that \( h \rightarrow 0 \), i.e. the water becomes shallow. At a fixed \( r_1 \), the scaled liquid depth \( h \) tends to zero as \( \theta_0 \rightarrow \pi/2 \).

Linear sloshing in a motionless tank is governed by the following boundary value problem (Lukovsky et al., 1984; Ibrahim, 2005):

\[
\Delta \phi = 0 \text{ in } Q_0; \quad \frac{\partial \phi}{\partial x} = \frac{\partial f}{\partial t} + \frac{\partial \phi}{\partial t} + g \frac{\partial f}{\partial y} = 0 \text{ on } \Sigma_0; \quad \frac{\partial \phi}{\partial n} = 0 \text{ on } S; \quad \int_{\Sigma_0} \frac{\partial \phi}{\partial x} \, dS = 0, \tag{1}
\]

where \( \phi(x,y,z,t) \) is the velocity potential, \( x = f(y,z,t) \) describes the free surface, \( n \) is the outer normal to \( S \) and \( g \) is the gravity acceleration scaled by \( r_0/g := g/r_0 \). The initial conditions:

\[
f(y,z,t_0) = F_0(y,z); \quad \frac{\partial f}{\partial t}(y,z,t_0) = F_1(y,z) \tag{2}
\]

at an instant time \( t = t_0 \), determine a unique solution of equation (1). The functions \( F_0 \) and \( F_1 \) define initial displacements of the free surface and its velocity, respectively.

### 2.2 Natural sloshing modes

The solution of equation (1) is associated with the free-standing waves:

\[
\phi(x,y,z,t) = \psi(x,y,z) \exp(i\omega t), \quad i^2 = -1 \tag{3}
\]

where \( \omega \) is the natural sloshing frequency. Inserting equation (3) into equation (1) leads to the spectral problem:

\[
\Delta \psi = 0 \text{ in } Q_0; \quad \frac{\partial \psi}{\partial x} = \kappa \psi \text{ on } \Sigma_0; \quad \frac{\partial \psi}{\partial n} = 0 \text{ on } S; \quad \int_{\Sigma_0} \frac{\partial \psi}{\partial x} \, dS = 0 \tag{4}
\]
where the eigenvalue $\kappa$ is defined by:

$$\kappa = \frac{\sigma^2}{g}. \quad (5)$$

The spectral problem (4) has a real positive point wise spectrum (Morand and Ohayon, 1995):

$$0 < \kappa_1 \leq \kappa_2 \leq \ldots \leq \kappa_n \leq \ldots$$

with a unique limit point at infinity, i.e. $\kappa_n \to \infty, n \to \infty$. Together with the constant function, projections of the eigenfunctions, $f_n(y,z) = \psi_n | \sum_0 \psi$ constitute an orthogonal basis in the mean-squares metrics. Because the velocity potential $\phi$ satisfies the volume conservation condition (see, the last integral equality in equation (1)), getting known $\{\kappa_n\}$ and $\{\psi_n\}$, one can represent the solution of equation (1) and equation (2) as a Fourier series by $\phi_n = \psi_n(x,y,z)\exp(i\sigma_n t)$, namely, as a superposition of the free-standing waves.

2.3 On the Ritz-Treftz schem for the spectral problem (4)

Problem (4) admits a minimax variational formulation (Feschenko et al., 1969; Morand and Ohayon, 1995), which is based on the positive functional:

$$K(\psi) = \frac{\int Q_0 (\nabla \psi)^2 dQ}{\int \sum_0 \psi^2 dS} \quad (6)$$

under the supplementary condition $\int Q_0 \psi dS = 0$. In that case, the absolute minimum of the functional equation (6) coincides with the lowest eigenvalue of the spectral problem (4). Furthermore, the necessary condition for an extrema of equation (6) leads to the variational equation:

$$\int Q_0 (\nabla \psi, \nabla \eta) dQ - \kappa \int \sum_0 \psi \eta dS = 0 \quad (7)$$

with respect to a non-constant function $\psi$, where $\eta$ is a smooth test-function.

Variational problem (7) may be solved by the Ritz-Treftz variational scheme. Approximate solutions are then posed as the following linear combination of smooth harmonic functions:

$$\psi(x,y,z) = \sum_{k=1}^{q} a_k B_k(x,y,z). \quad (8)$$

Substituting equation (8) into equation (7) and using $B_l(i = 1,\ldots,q)$ as test-functions, one obtains the spectral matrix problem:

$$\sum_{k=1}^{q} (\{\alpha_{ik}\} - k \{\beta_{ik}\}) a_k = 0, \quad i = 1, \ldots, q \quad (9)$$

Here, the elements of the non-negative matrices $A = \{\alpha_{ik}\}$, $B = \{\beta_{ik}\}$ are computed by the formulae:
\[
\alpha_{ik} = \int_{Q_0} (\nabla B_i, \nabla B_k) dQ = \int_{\sum_{i} S} \frac{\partial B_i}{\partial n} B_k dS, \quad \beta_{ik} = \int_{\sum_{i} S} B_i B_k dS, \tag{10}
\]

and the approximate eigenvalues are roots of the equation:

\[
\det(A - \kappa B) = 0, \tag{11}
\]

which appears as the necessary solvability condition of system (9). By increasing the dimension \( q \) the non-zero roots of equation (11) converge (from above) to the lower eigenvalues of equation (4). Approximate eigenfunctions (8) are formed by the eigenvectors of equation equation (9), \( \{ a_{ik} \mid k = 1, \ldots, q \} \).

A key difficulty of the Ritz-Trefftz scheme consists of establishing a suitable analytical functional basis \( \{ B_k \} \). The completeness of functional sets significantly depends on the actual shape of \( Q_0 \). To the author’s knowledge, there are two types of analytical functional sets, which may be adapted to the studied case. The first set is proposed in the book by Lukovsky \textit{et al.} (1984). It follows from a separation of variables in the Laplace equation done in the spherical coordinate system. These harmonic solutions are of polynomial structure (harmonic polynomial solutions, HPS) in the Cartesian coordinate system. Their completeness is proved for all star-shaped domains with respect to the origin. Being rewritten in a cylindrical coordinate system, the HPS admit the separation of the angular coordinate and keep polynomial structure with regard to the remaining coordinates, i.e. in projections on a meridional cross-section. A specific functional basis (SFB) is presented by Gavrilyuk \textit{et al.} (2005). It is derived in the cylindrical coordinate system combined with a non-conformal mapping of the meridional cross-section. The functional set is harmonic and satisfies the zero-Neumann condition on the conical walls. Furthermore, we utilise these two functional sets in the Ritz-Trefftz scheme to solve the spectral problem (4). The three-dimensional problem and its variational formulation (7) are thereby reduced to two dimensions by separating the angular-type variable.

3. Ritz-Trefftz method based on the HPS

We use the cylindrical coordinate system \((X, \xi, \eta)\) linked with the original Cartesian coordinates by:

\[
x = X + X_0, \quad y = \xi \cos \eta, \quad z = \xi \sin \eta. \tag{12}
\]

Here, the lag \( X_0 \) along the vertical axis is introduced to superpose the origin of the cylindrical coordinate system with the waterplane. The solution of equation (4) is represented in the following form:

\[
\psi(X, \xi, \eta) = \varphi_m(X, \xi) \begin{pmatrix} \sin m \eta \\ \cos m \eta \end{pmatrix}, \quad m = 0, 1, 2, \ldots \tag{13}
\]

This makes it possible to separate the angular coordinate \( \eta \) and to reduce the three-dimensional boundary value problem (4) to the following \( m \)-parametric family \((m \) is a non-negative integer) of two-dimensional spectral problems:
\[
\frac{\partial}{\partial X} \left( \xi \frac{\partial \varphi_m}{\partial X} \right) + \frac{\partial}{\partial \xi} \left( \xi \frac{\partial \varphi_m}{\partial \xi} \right) - \frac{m^2}{\xi} \varphi_m = 0 \text{ in } G; \quad \frac{\partial \varphi_m}{\partial X} = \kappa_m \varphi_m \text{ on } L_0; \\
\frac{\partial \varphi_m}{\partial n} = 0 \text{ on } L; \quad |\varphi_m(X, 0)| < \infty; \quad \int_{L_0} \xi \frac{\partial \varphi_0}{\partial X} \, d\xi = 0.
\]

Problem (14) is defined in a meridional plane of \(Q_0\) and \(L = L_1 + L_2\) (Figure 2). This means that the eigenvalues of the original three-dimensional problem constitute a two-parametric set \(\kappa = \kappa_m(m = 0, 1, \ldots; i = 1, 2, \ldots)\), where \(i \geq 1\) enumerates the eigenvalues of equation (14) in ascending order. The corresponding eigenfunctions of equation (13) take the form equation (13) with \(\varphi_m = \varphi_m(X, \xi)\).

According to Lukovsky et al. (1984), the HPS admit the following form in the meridional plane (after separation of the \(\eta\)-coordinate):

\[
w^{(m)}(X, \xi) = \frac{2(k - m)!}{(k + m)!} R_k P^{(m)}(\frac{X}{R}), k \geq m, \quad R = \sqrt{X^2 + \xi^2},
\]

Where \(P^{(m)}\) are Legendre’s functions of first kind. The functions \(\{w^{(m)}\}\) are the solutions of the first equation of (14). It can be shown that \(w^{(m)}\) has indeed a polynomial structure in terms of \(X\) and \(\xi\). The first functions of the set equation (15) take the form:

\[
w_0^{(0)} = 1, \quad w_1^{(0)} = X, \quad w_2^{(0)} = X^2 - \frac{\xi^2}{2}, \quad \ldots \quad (m = 0),
\]
\[
w_1^{(1)} = \xi, \quad w_2^{(1)} = X \xi, \quad w_3^{(1)} = X^2 \xi - \frac{\xi^3}{4}, \quad \ldots \quad (m = 1),
\]
\[
w_2^{(2)} = \xi^2, \quad w_3^{(2)} = X \xi^2, \quad w_4^{(2)} = X^2 \xi^2 - \frac{\xi^4}{6}, \quad \ldots \quad (m = 2),
\]
\[
w_3^{(3)} = \xi^3, \quad w_4^{(3)} = X \xi^3, \quad w_5^{(3)} = X^2 \xi^3 - \frac{\xi^6}{8}, \quad \ldots \quad (m = 3).
\]

The computation of \(\{w^{(m)}\}\) can be realised by the following recurrence relations:
\[ \frac{\partial w_{k}^{(m)}}{\partial X} = (k - m)w_{k-1}^{(m)}; \quad \xi \frac{\partial w_{k}^{(m)}}{\partial \xi} = kw_{k}^{(m)} - (k - m)Xw_{k-1}^{(m)}, \]
\[
(k - m + 1)w_{k+1}^{(m)} + (2k + 1)Xw_{k}^{(m)} - (k - m)(X^2 + \xi^2)w_{k-1}^{(m)},
\]
\[
(k - m + 1)\xi w_{k}^{(m+1)} = 2(m + 1)((X^2 + \xi^2)w_{k}^{(m)} - Xw_{k-1}^{(m)}).
\]

By separating the \( \eta \)-coordinate in the variational formulation (7) and in the representation equation (8), we arrive at the following \( m \)-parametric families (\( m \) is non-negative integer) of approximate solutions:
\[
\varphi_m(X, \xi) = \sum_{k=1}^{q} a_k^{(m)} w_{k+m-1}^{(m)}(X, \xi),
\]
and spectral matrix problems:
\[
\sum_{k=1}^{q} \left( \{ \alpha_{ik}^{(m)} \} - \kappa_m \{ \beta_{ik}^{(m)} \} \right) a_k = 0 \quad (i = 1, \ldots, q),
\]
\[
\text{det} \left( \{ \alpha_{ik}^{(m)} \} - \kappa_m \{ \beta_{ik}^{(m)} \} \right) = 0,
\]
following from (9). The elements \( \{ \alpha_{ik}^{(m)} \} \) and \( \{ \beta_{ik}^{(m)} \} \) are computed for the \( \Lambda \)-cones by the formulae:
\[
\alpha_{ij}^{(m)} = \int_{0}^{r_1} \left( \xi \frac{\partial w_{i+m-1}^{(m)} w_{j+m-1}^{(m)}}{\partial X} \right)_{X=0} + \int_{-h}^{0} \left( \xi \frac{\partial w_{i+m-1}^{(m)} w_{j+m-1}^{(m)}}{\partial \xi} \right)_{\xi=\tan \theta \lambda X-r_1} \text{d}X
\]
\[
- \tan \theta_0 \int_{-h}^{0} \left( \xi \frac{\partial w_{i+m-1}^{(m)} w_{j+m-1}^{(m)}}{\partial X} \right)_{\xi=\tan \theta_0 X-r_1} \text{d}X - \int_{0}^{1} \left( \xi \frac{\partial w_{i+m-1}^{(m)} w_{j+m-1}^{(m)}}{\partial X} \right)_{X=-h}\text{d}\xi,
\]
\[
\beta_{ij}^{(m)} = \int_{0}^{r_1} \left( \xi w_{i+m-1}^{(m)} w_{j+m-1}^{(m)} \right)_{X=0} \text{d}\xi,
\]
and for the \( V \) cones by the formulae:
\[
\alpha_{ij}^{(m)} = \int_{0}^{1} \left( \xi \frac{\partial w_{i+m-1}^{(m)} w_{j+m-1}^{(m)}}{\partial X} \right)_{X=0} + \int_{-h}^{0} \left( \xi \frac{\partial w_{i+m-1}^{(m)} w_{j+m-1}^{(m)}}{\partial \xi} \right)_{\xi=\tan \theta \lambda X+1} \text{d}X
\]
\[
- \tan \theta_0 \int_{-h}^{0} \left( \xi \frac{\partial w_{i+m-1}^{(m)} w_{j+m-1}^{(m)}}{\partial X} \right)_{\xi=\tan \theta_0 X+1} \text{d}X - \int_{0}^{r_1} \left( \xi \frac{\partial w_{i+m-1}^{(m)} w_{j+m-1}^{(m)}}{\partial X} \right)_{X=-h}\text{d}\xi,
\]
\[
\beta_{ij}^{(m)} = \int_{0}^{r_1} \left( \xi w_{i+m-1}^{(m)} w_{j+m-1}^{(m)} \right)_{X=0} \text{d}\xi.
\]
For each fixed $m$, the second equation of (17) has $q$ positive roots ($n = 1, 2, \ldots, q$). The multiplicity should be accounted for. Since the Ritz-Treftz method implies the minimisation of a functional, the approximate values $k_{mn}$ converge from above. This makes it possible to check the convergence by the number of significant digits, which do not change as increases. The method gives the best approximation for the lowest eigenvalue $\kappa_{m1}$.

Convergence. Our numerical experiments were primarily dedicated to eigenvalues $\kappa_{m1}$, $m = 0, 1, 2, 3$. These eigenvalues are responsible for the lowest natural modes, which give a decisive contribution to hydrodynamic loads (Ibrahim, 2005; Gavrilyuk et al., 2005). In the case of V-tanks, the method shows a fast convergence to $\kappa_{m1}$ and provides satisfactory accuracy for $\kappa_{m2}$ and $\kappa_{m3}$ too. It shows a slower convergence for the \( \Lambda \)-tanks. Furthermore, the convergence depends not only on the tank type (V or \( \Lambda \)-shaped), but also on the semi-apex angle $\theta_0$ and the dimensionless parameter $0 < r_1 < 1$. The results in Table I (A) exhibit a typical convergence behaviour for the V-tanks with $10^\circ \leq \theta_0 \leq 75^\circ$ and $0.2 \leq r_1 \leq 0.9$. The table shows stabilisation of 5-6 significant digits as $q \geq 14$. The best accuracy is established for the lowest spectral parameter $\kappa_{11}$. The accuracy grows with $q$ for $m \neq 0$. However, computations of the value $K_{01}$, which is responsible for the axial-symmetric natural mode, may become unstable for $q > 17$. This explains why we do not present numerical results on this eigenvalue for $q = 20$ While $m \neq 1$ and $0.2 \leq r_1 \leq 0.9$, increasing $\theta_0 > 75^\circ$ leads to a slower convergence. In this case, $q = 17, \ldots, 20$ guarantees only 3-4 significant digits (necessary engineering accuracy). The same number of basic functions keeps this number of significant digits for the tanks with $r_1 < 0.2$ and $10^\circ \leq \theta_0 \leq 75^\circ$. When $r_1$ tends to zero (truncated tank is close to a non-truncated one), the approximations $\kappa_{11}$ were validated by numerical results reported by Gavrilyuk et al. (2005) as well as by experimental data given in Bauer (1982).

In Table I (B), typical convergence behaviour for the \( \Lambda \)-shaped tanks with $10^\circ \leq \theta_0 \leq 75^\circ$ is presented. A comparative analysis of parts (A) and (B) illustrates that the method is in the latter case less efficient. In particular, computations of $K_{01}$ are not so precise. Moreover, 18-20 basic functions lead to 4-5 significant digits of $\kappa_{m1}$ only for $r_1 \geq 0.4$. This is not the case for lower values of $r_1$. Furthermore, when $r_1 \leq 0.2, q = 17, \ldots, 20$ can only guarantee 2-3 significant digits for $\kappa_{11}$. The slower convergence for the \( \Lambda \)-shaped tanks can in part be clarified by the occurrence of singular first derivatives of the eigenfunctions $\psi_m$ at an inner vertex between $L_0$ and $L_1$ (see the mathematical results by Lukovsky et al., 1984). The HPS are smooth in the $(\xi, X)$-plane, and, therefore, do not capture this singular behaviour. The singularity disappears when the corner angle is less than $90^\circ$. This occurs only for the \( \Lambda \)-shaped tanks. The V-shaped tanks are characterised by a similar singularity at the vertex formed by $L_1$ and $L_2$. However, because the natural modes (eigenfunctions $\psi_m$) should “decay” exponentially downward, the method may be sensitive with respect to that singularity only for shallow water. Our numerical experiments confirm this fact as $r_1 < 0.1$.

3.1 Ritz-Treftz method based on the SFB

The nonlinear resonant sloshing is effectively studied by multimodal methods. As it is shown by Lukovsky (1975), Lukovsky and Timokha (2002), these methods require analytical expressions for the natural modes, which:
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<tr>
<td>6</td>
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<td>3.385599</td>
<td>3.381822</td>
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<td>2.197162</td>
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<td>10</td>
<td>27.64904</td>
<td>10</td>
<td>101.0621</td>
</tr>
</tbody>
</table>

Notes: Column (A) is for a V-shaped tank, (B) corresponds to a A-shaped tank, \( \theta_0 = 30^\circ \)
are analytically expandable over the waterplane;
• satisfy a zero-Neumann condition at the tank walls and, if the tank is non-cylindrical; and
• can be transformed to a curvilinear coordinate system \((x_1, x_2, x_3)\), in which the free surface is governed by the normal form representation \(x_1 = f(x_2, x_3, t)\).

An example of suitable approximate natural modes for non-truncated V-tanks is given by Lukovsky (1990) and Gavrilyuk et al. (2005). The present section generalises these results.

**Curvilinear coordinate system.** The non-Cartesian parametrisation proposed by Gavrilyuk et al (2005) links the \(x,y,z\) coordinates with \(x_1, x_2, x_3\) as follows:

\[
x = x_1, \quad y = x_1 x_2 \cos x_3, \quad z = x_1 x_2 \sin x_3.
\] (18)

Thus, the variable \(x_3 = \eta\) is the polar angle in the \(Oyz\)-plane and corresponds to \(\eta\) in equation (12). Figure 3 demonstrates that the hydrostatic liquid domain \(Q_0\) takes in the \((x_1, x_2, x_3)\)-system the form of an upright rectangular base cylinder \((x_0 \leq x_1 \leq x_{10}, 0 \leq x_2 \leq x_{20}, 0 \leq x_3 \leq 2\pi)\). The domain \(G^*\) represents a rectangle with the sides \(h = x_{10} - x_0\) and \(x_{20} = \tan \theta_0\) in the \(Ox_2x_1\)-plane. Here, the radius of the undisturbed water plane is \(r_t = 1\) for the V-tanks and \(r_t = r_1\) for the L-tanks. Having presented:

\[
\varphi(x_1, x_2, x_3) = \psi_m(x_1, x_2) \begin{pmatrix} \sin m x_3 \\ \cos m x_3 \end{pmatrix}, \quad m = 0, 1, 2, \ldots
\] (19)

and following Gavrilyuk et al. (2005), one obtains that the original three-dimensional problem (4) admits separation of the spatial variable \(x_3\). Furthermore, the transformation (18) generates the following \(m\)-parametric family of spectral problems with respect to:

\[
p \frac{\partial^2 \psi_m}{\partial x_1^2} + 2q \frac{\partial^2 \psi_m}{\partial x_1 \partial x_2} + s \frac{\partial^2 \psi_m}{\partial x_2^2} + d \frac{\partial \psi_m}{\partial x_2} - m^2 c \psi_m = 0 \quad \text{in} \quad G^*,
\] (20)

\[
p \frac{\partial \psi_m}{\partial x_1} + q \frac{\partial \psi_m}{\partial x_2} = \kappa_m p \psi_m \quad \text{on} \quad L^*_0,
\] (21)
\[ s \frac{\partial \psi_m}{\partial x_2} + q \frac{\partial \psi_m}{\partial x_1} = 0 \quad \text{on } \mathcal{L}_1^*, \]  
(22)  

\[ p \frac{\partial \psi_m}{\partial x_1} + q \frac{\partial \psi_m}{\partial x_2} = 0 \quad \text{on } \mathcal{L}_2^*, \]  
(23)  

\[ |\psi_m(x_1, 0)| < \infty, \quad m = 0, 1, 2, \ldots, \]  
(24)  

\[ \int_{x_20}^{x_2} \psi_0 x_2 dx_2 = 0, \]  
(25)  

where \( G^* = \{(x_1, x_2): x_0 \leq x_1 \leq x_{10}, 0 \leq x_2 \leq x_{20}\}, \) \( \bar{W}^{(0)}, \) \( d = 1 + 2x_2^2, c = 1/x_2 \) and the boundary of \( G^* \) consists of the portions \( \mathcal{L}_0^*, \mathcal{L}_1^* \) and \( \mathcal{L}_2^* \).

**Particular solutions of equation (20)** and equation (22), Gavrilyuk et al. (2005) studied a spectral problem, which is similar to equations (20)-(25). Following results by Eisenhart, they proved that equation (20) and equation (22) allow together for the separation of the spatial variables \( x_1 \) and \( x_2 \). For our problem, this separation leads to the following particular solutions:

\[ x_1^n T^{(m)}(x_2) \quad \text{and} \quad \frac{T^{(m)}(x_2)}{x_1^n}, \quad \nu \geq 0. \]  
(26)  

In order to determine \( T^{(m)}(x_2) \), we have to consider the following homogeneous boundary value problem, which depends on the real parameter \( \nu \):

\[ x_2^2(1 + x_2^2)T^{(m)}(x_2) + x_2(1 + 2x_2^2 - 2x_2^2)T^{(m)}(x_2) + [\nu(\nu-1)x_2^2 - m^2]T^{(m)} = 0, \]  
(27)  

\[ T^{(m)}(x_20) = \nu \frac{x_20}{1 + x_20^2} T^{(m)}(x_20), \quad |T^{(m)}_\nu(0)| < \infty. \]  
(28)  

It can be shown that the problem (27) and (28) has only nontrivial solutions for a countable set of values \( \nu = \nu_{mn} > 0 (m = 0, 1, \ldots; n = 1, 2, \ldots) \).

The second class of functions, \( T^{(m)}(x_2) \), appears only in the case of \( x_0 \neq 0 \), i.e. when the conical tank is truncated. Computation of \( T^{(m)}(x_2) \) leads to the following \( \nu \)-parametric problem:

\[ x_2^2(1 + x_2^2)\bar{T}^{(m)} + x_2(1 + 4x_2^2 - 2x_2^2)\bar{T}^{(m)} + [\nu(\nu+1)x_2^2 - m^2]\bar{T}^{(m)} = 0, \]  
(29)  

\[ \bar{T}^{(m)}(x_20) + (\nu+1) \frac{x_20}{1 + x_20^2} \bar{T}^{(m)}(x_20) = 0. \]  
(30)  

Obviously, nontrivial solutions of equation (29) and equation (30) exist only for a countable set of nonnegative values \( \nu \).

Let us now show that the solution of equations (27) and (29) can be expressed in terms of the spheroidal harmonics and the set \( \{\nu_{mn}\} \) is the same for the problems (27) and (28) and the problems (29) and (30). For this purpose, we change the variables in equations (27) and (29) by \( \mu = (1 + x_2^2)^{-1/2} \) and substitute \( y(\mu) = \mu^\nu T(\mu) \) and
\( y(\mu) = \mu^{-1-\nu} T(\mu) \) into equation (27) and equation (29), respectively. This reduces the two equations to the same well-known differential equation:

\[
(1 - \mu^2)y''(\mu) - 2\mu y'(\mu) + \left[ \nu(\nu + 1) - \frac{m^2}{1 - \mu^2} \right] y(\mu) = 0,
\]

whose solutions coincide with the Legendre function of first kind, i.e. \( y(\mu) = P^m_\nu(\mu) \).

Furthermore, treating the boundary conditions (28) and (30) in the same way and using the substitution \( \mu = \cos \theta \), the following common equation is obtained:

\[
\frac{\partial P^m_\nu(\cos \theta)}{\partial \theta} \bigg|_{\theta=\theta_0} = 0. \tag{31}
\]

This equation can be considered as a transcendental equation for the computation of the values \( \{ \nu_{mn} \} \). Appendix (Figure A1) presents the first 12 values of \( \{ \nu_{mn} \} \) \( (m = 0,1,2,3) \) versus \( \theta_0 \).

In conclusion, with the technique described above, we get the following nontrivial particular solutions

\[
T_{\nu_{mk}}^{(m)}(x_2) = (1 + x_2^0)^{\frac{\nu_{mk}}{2}} P^m_{\nu_{mk}} \left( \frac{1}{\sqrt{1 + x_2^0}} \right), \tag{32}
\]

\[
\frac{\partial P^m_\nu(\cos \theta)}{\partial \theta} \bigg|_{\theta=\theta_0} = 0. \tag{31}
\]

Particular solutions as a functional basis. Let the particular solutions (26), (32) and (33) be rewritten in the form:

\[
W_{k}^{\nu_{mk}}(x_1,x_2) = N_{k}^{\nu_{mk}} x_1^{\nu_{mk}} T_{\nu_{mk}}^{\nu_{mk}}(x_2), \quad \bar{W}_{k}^{\nu_{mk}}(x_1,x_2) = \bar{N}_{k}^{\nu_{mk}} x_1^{1-\nu_{mk}} T_{\nu_{mk}}^{\nu_{mk}}(x_2). \tag{34}
\]

Here, \( N_{k}^{\nu_{mk}} \) and \( \bar{N}_{k}^{\nu_{mk}} \) are multipliers which are chosen to satisfy the following condition:

\[
1 = \| W_{k}^{\nu_{mk}} \|_{L_2^* + L_0^*}^2 = \| \bar{W}_{k}^{\nu_{mk}} \|_{L_2^* + L_0^*}^2
\]

\[
= \int_0^{x_{20}} x_2 [(W_{k}^{\nu_{mk}} |_{x_1 = x_{10}})^2 + (\bar{W}_{k}^{\nu_{mk}} |_{x_1 = x_{10}})^2] dx_2
\]

\[
= \int_0^{x_{20}} x_2 [((\bar{W}_{k}^{\nu_{mk}} |_{x_1 = x_{10}})^2 + ((\bar{W}_{k}^{\nu_{mk}} |_{x_1 = x_{10}})^2] dx_2. \tag{35}
\]

equation (35) says that \( W_{k}^{\nu_{mk}} \) and \( \bar{W}_{k}^{\nu_{mk}} \) have the unit norm (in the mean-squares metrics) on the boundary \( L_2^* + L_0^* \), where equations (21) and (23) should be approximately satisfied. Explicit formulae for these normalising multipliers have the form:
This implies a re-definition of the functions $W_{k}^{(0)}$ and $\tilde{W}_{k}^{(0)}$ by: $W_{k}^{(0)} := W_{k}^{(0)} - c_{k}^{(0)}$, $\tilde{W}_{k}^{(0)} := \tilde{W}_{k}^{(0)} - \tilde{c}_{k}^{(0)}$, where:

$$c_{k}^{(0)} = \frac{2}{x_{20}^{2}} \int_{0}^{x_{20}} x_{2} W_{k}^{(0)}(x_{10}, x_{2}) \, dx_{2}, \quad \tilde{c}_{k}^{(0)} = \frac{2}{x_{20}^{2}} \int_{0}^{x_{20}} x_{2} \tilde{W}_{k}^{(0)}(x_{10}, x_{2}) \, dx_{2}.$$

**Variational method.** In accordance with the Ritz-Treftz scheme, we represent approximate solutions of equations (20)-(25) in the form:

$$\psi_{m}(x_{1}, x_{2}) = \sum_{k=1}^{q_{1}} a_{k}^{(m)} W_{k}^{(m)} + \sum_{l=1}^{q_{2}} \tilde{a}_{l}^{(m)} \tilde{W}_{l}^{(m)}.$$  \hspace{1cm} (36)

By separating the $x_{3}$ coordinate in variational formulation (7) (after substitution of equation (19)), representation (36) leads to the $m$-parametric family of spectral problems:

$$\sum_{k=1}^{Q} \left( \left\{ \alpha_{m}^{(m)} \right\}_j - \kappa_{m} \left\{ \beta_{m}^{(m)} \right\}_j \right) \alpha_{k} = 0 \quad (i = 1, \ldots, Q),$$  \hspace{1cm} (37)

$$\det \left( \left\{ \alpha_{m}^{(m)} \right\}_i - \kappa_{m} \left\{ \beta_{m}^{(m)} \right\}_i \right) = 0.$$  

The spectral problem (37) has $Q = q_{1} + q_{2}$ eigenvalues. Because the representation (36) contains two types of functions, namely $W_{k}^{(m)}$ and $\tilde{W}_{l}^{(m)}$, there exist four sub-matrices of $\left\{ \alpha_{m}^{(m)} \right\}_j$ and $\left\{ \beta_{m}^{(m)} \right\}_j$ such that:

$$\alpha_{m}^{(m)} = \begin{pmatrix} \alpha_{i1}^{(m)} & \alpha_{i2}^{(m)} \\ \alpha_{i3}^{(m)} & \alpha_{i4}^{(m)} \end{pmatrix}, \quad \beta_{m}^{(m)} = \begin{pmatrix} \beta_{j1}^{(m)} & \beta_{j2}^{(m)} \\ \beta_{j3}^{(m)} & \beta_{j4}^{(m)} \end{pmatrix}.$$  

The elements $\left\{ \alpha_{ijs}^{(m)} \right\}_j$ and $\left\{ \beta_{ijs}^{(m)} \right\}_j$, $s = 1, \ldots, 4$, are computed by the formulae:

$$\alpha_{i1}^{(m)} = \int_{0}^{x_{20}} \left( x_{1}^{2} x_{2} \frac{\partial W_{i}^{(m)}}{\partial x_{1}} - x_{1} x_{2}^{2} \frac{\partial W_{i}^{(m)}}{\partial x_{2}} \right)_{x_{2} = h_{i}} \, dx_{2},$$

$$-\int_{0}^{x_{20}} \left( x_{1}^{2} x_{2} \frac{\partial W_{j}^{(m)}}{\partial x_{1}} - x_{1} x_{2}^{2} \frac{\partial W_{j}^{(m)}}{\partial x_{2}} \right)_{x_{2} = h_{j}} \, dx_{2},$$

Natural sloshing frequencies
\[ \alpha_{ij}^{(m)} = \int_0^{x_0} \left( x_1^2 \frac{\partial W_i^{(m)}}{\partial x_1} - x_1 x_2^2 \frac{\partial W_i^{(m)}}{\partial x_2} \right) x_1 = h_i \ W_i^{(m)} \ dx_2 \]

\[ - \int_0^{x_0} \left( x_1^2 x_2 \frac{\partial W_i^{(m)}}{\partial x_1} - x_1 x_2^2 \frac{\partial W_i^{(m)}}{\partial x_2} \right) x_1 = h_i \ W_j^{(m)} \ dx_2, \]

\[ \alpha_{ij}^{(m)} = \int_0^{x_0} \left( x_1^2 \frac{\partial W_j^{(m)}}{\partial x_1} - x_1 x_2^2 \frac{\partial W_j^{(m)}}{\partial x_2} \right) x_1 = h_i \ W_j^{(m)} \ dx_2 \]

\[ - \int_0^{x_0} \left( x_1^2 x_2 \frac{\partial W_j^{(m)}}{\partial x_1} - x_1 x_2^2 \frac{\partial W_j^{(m)}}{\partial x_2} \right) x_1 = h_i \ W_j^{(m)} \ dx_2, \]

\[ \beta_{ij}^{(m)} = h_i^2 \int_0^{x_0} x_2 \left( W_i^{(m)} W_j^{(m)} \right) x_1 = h_i \ dx_2, \quad \beta_{ij}^{(m)} = h_i^2 \int_0^{x_0} x_2 \left( W_i^{(m)} W_j^{(m)} \right) x_1 = h_i \ dx_2, \]

\[ \beta_{ij}^{(m)} = h_i^2 \int_0^{x_0} x_2 \left( W_i^{(m)} W_j^{(m)} \right) x_1 = h_i \ dx_2, \quad \beta_{ij}^{(m)} = h_i^2 \int_0^{x_0} x_2 \left( W_i^{(m)} W_j^{(m)} \right) x_1 = h_i \ dx_2. \]

In the case of \( \Lambda \)- and V-tanks, we have \( h_i = r_1 / \tan \theta_0, h_b = 1 / \tan \theta_0 \) and \( h_i = 1 / \tan \theta_0, h_b = r_1 / \tan \theta_0 \), respectively.

**Convergence.** Column A in Table II shows a typical convergence behaviour in the case of V-tanks with \( 10^\circ \leq \theta_0 \leq 75^\circ \) and \( 0.2 \leq r_1 \leq 0.9 \). When \( 0.2 \leq r_1 \leq 0.55 \), the method generates 4-5 significant digits of \( \kappa_{m1} \) for \( q = q_1 = q_2 = 7, \ldots, 10 \) (14, \ldots, 20 basic functions). This is consistent with the convergence results in Section 3. However, the SFB keeps also a fast convergence to \( \kappa_{m1} \) for \( r_1 < 0.2 \). This includes the case of \( \kappa_{01} \), which has not been satisfactorily handled by the HPS. Moreover, when \( 0 < r_1 \leq 0.4 \), the number of significant digits is larger (for the same number of basic functions) for \( 15^\circ \leq \theta_0 < 30^\circ \), but it is only marginally smaller for \( 15^\circ \leq \theta_0 \). Gavrilyuk et al. (2005) related such a slower convergence of an analogous method for smaller semi-apex angles to the asymptotic behaviour of the exact solution along the vertical axis. Their conclusion is that the eigenfunctions \( \psi_m \) should exponentially decay downward \( Ox \) for a circular cylindrical tank, to which the conical domain tends as \( \theta_0 \) decreases. However, \( W_k^{(m)} \) and \( \overline{W}_k^{(m)} \) do not capture this decaying. Furthermore, decreasing the dimensionless liquid depth \( h(r_1 \rightarrow 1 \text{ or } \theta_0 \rightarrow 90^\circ) \) may cause a lower accuracy (3-4 significant digits for 18-24 basic functions).
<table>
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<th>$Q$</th>
<th>$r_1 = 0.2$</th>
<th>$r_1 = 0.4$</th>
<th>$r_1 = 0.6$</th>
<th>$r_1 = 0.8$</th>
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</tr>
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</table>

**Notes:** Column (A) is for a V-shaped tank; (B) corresponds to a Λ-shaped tank, $\theta_0 = 30^\circ$.

### Table II.

**Convergence to $\kappa_{nl}$,**

$m = 0, 1, 2, 3$, for different $r_1$ versus the number of basic functions $q = q_1 = q_2$ in (36)
Column B in Table II shows convergence for a \( \Lambda \)-tank. The same \( r_1 \) and \( \theta_0 \) as in the column A are chosen. It can be seen that the numerical results may be less precise than those in Section 3. For instance, the same number of basic functions gives only 2-3 significant digits when \( 0.2 \leq r_1 \leq 0.9 \). However, in contrast to the HPS, the SFB provides reliable computations for the case of an axial symmetric mode \( \kappa_{01} \). In addition, whereas \( 0.05 \leq r_1 \leq 0.4 \), the lowest eigenvalue \( \kappa_{11} \) is calculated with a better accuracy. For the same \( r_1 \), increasing the semi-apex angle may lead to a slower convergence. If \( q_1 = q_2 = 12 \), the number of significant digits also decreases as \( r_1 \rightarrow 1 \). This “shallow water” case is handled with 2-3 significant digits as \( q_1 = q_2 = \ldots, 14 \).

The presence of the two types of basic functions in the representation (36) makes it possible to vary \( q_1 \) and \( q_2 \) to obtain a better approximation with the same total number of basic functions \( Q = q_1 + q_2 \). Variations of \( q_1 \) and \( q_2 \) with a fixed \( Q \geq 16 \) showed that a better accuracy of \( \kappa_{m1} \) can be expected for \( q_2 > q_1 \). In particular, this is true for smaller liquid depths. For example, when the V-shaped tank is characterised by \( \theta_0 = 30^\circ \) and \( r_1 = 0.9 \), the approximate \( \kappa_{11} = 54233738 \) can be obtained with either \( q_1 = q_2 = 12(Q = 24) \) or \( q_1 = 7, q_2 = 12(Q = 19) \).

4. Comparative analysis of the two method
A comparison of numerical experiments performed with the two different functional bases shows that the method based on the HPS is more accurate for smaller liquid depths \( (0.6 \leq r_1) \). However, larger liquid depths \( (r_1 \leq 0.4) \) are better treated with the second method. This can clearly be seen for the \( \Lambda \)-shaped tanks: the calculations with the method by the SFB keep robustness as the number of basic functions is increased, while the first method fails for larger dimensions. Generally speaking, the accuracy of both methods is similar only for V-tanks with \( 0.2 \leq r_1 \leq 0.55 \), \( \theta_0 > 10^0 \).

Even though the number of basic functions is small, the two proposed methods give an accurate approximation of the lowest eigenvalue \( \kappa_{11} \). The lowest eigenvalue determines the lowest natural frequency by \( \sigma_{11} = \sqrt{g \kappa_{11}} \). This frequency is of primary interest for modelling tower vibrations with a V-shaped tank. Therefore, we placed special emphasis on a comparison of the numerical results obtained by the two methods for \( \kappa_{11} \). The results are illustrated in Figure 4(a) and (b). Here, domains in the \((r_1, \theta_0)\)-plane are identified, for which each of the methods gives the same number of significant digits with twenty basic functions. One can see that the accuracy of the first method (HPS) may become low only for small \( \theta_0 \) and \( r_1 \), e.g. for large liquid depths. In the other cases, the method guarantees a fast convergence and high accuracy. On the other hand, small \( \theta_0 \)
and $r_1$ are satisfactory handled by the second method (SFB). However, this method converges slowly as $r_1 \to 1$ and $\theta_0 > 45^\circ$, e.g. for small liquid depths.

5. The lowest natural sloshing frequency

The natural sloshing frequencies $\sigma_{m1}$ are functions of the liquid depth $h$, the semi-apex angle $\theta_0$ and the radii $r_1$ and $r_0$. For the V-tanks, an increasing of $\theta_0$ decreases the non-dimensional eigenvalues $\kappa_{m1}$ (the non-dimensional natural sloshing frequencies $\sigma_{m1}^2 r_0 / g = \kappa_{m1}$). For the A-tanks, an increasing of $\theta_0$ increases $\kappa_{m1}$.

Dependence of $\kappa_{m1}$ on the ratio $r_1/r_0$ is illustrated in Figure 5(a) and (b). These demonstrate that the non-dimensional frequencies decrease as the ratio $r_1/r_0 = (1 + \tan(\theta_0) h/r_1)$ decreases. In term of the fixed dimensional values of $r_1$ and $\theta_0$, this means that the non-dimensional sloshing frequencies decrease with decreasing the liquid depth $h$.

One interesting fact is that $\kappa_{01} \approx \kappa_{21}$ for the studied geometric parameters in the case of A-tanks. This is the same as for an upright circular cylindrical tank.

![Figure 5](image-url)

**Figure 5.** Eigenvalues $\kappa_{m1}$ versus $r_1$ for A-shaped (Case a) and V-shaped tanks (Case b)
Further, the natural sloshing frequencies are very close to those for the non-truncated conical V-shape tank as $0 \leq r_1 < 0.6$. Truncation matters for $r_1 \rightarrow 1$, i.e. for shallow-water sloshing.

The natural sloshing frequency an $\sigma_{11}$ of practical importance for the design of water towers (Damatty and Sweedan, 2006). Having in mind this fact, we present in Table III the values of $\kappa_{11}$ versus $\theta_0$ and $r_1$. The corresponding computations have been done to guarantee up to five significant digits. The dimensional natural sloshing frequency $\sigma_{11}$ is computed from $\kappa_{11}$ by the formula:

$$\sigma_{11} = \sqrt[\sqrt{g\kappa_{11}(\theta_0, r_1/r_0)}}{r_0},$$

where $g$ and $r_1$ are not scaled by $r_0$. The numerical data from Table III can therefore be used in both the structural design and the validation of other numerical methods.

6. Concluding remarks
We have proposed two efficient numerical-analytical methods for the computation of the natural sloshing frequencies and modes in truncated conical tanks. These methods are based on the Ritz-Treftz variational scheme. Extensive numerical experiments showed that these methods have different domains of applicability in terms of the semi-apex angle, the liquid depth and the tank type (V- or Λ-shaped).

The methods may have slow convergence behaviour and even diverge for some Λ-shaped tanks. This fact can be explained by the singular behaviour of the natural modes at the contact line formed by the waterplane and the conical walls. From a mathematical point of view, if the smooth bases would be augmented by a harmonic function, which has the specified singular behaviour, the convergence can be considerably improved. Lukovsky et al. (1984) gave examples of such an augmented function for two-dimensional spectral sloshing problems. Dedicated mathematical studies are needed to specify a suitable singular function for our case.

Bauer (1982) and Dokuchaev (1964) compared the numerical data presented in this paper with simplified analytical approximation, which were obtained for pure conical tanks with small semi-apex angles. Satisfactory agreement was observed only for a small angle ($\theta_0 < 15^\circ$). This agreement is consistent with the assumptions used by Bauer (1982) who replaced the planar mean liquid surface by spheric segment.

In future work, an emphasis should be placed on the shallow water sloshing. This case requires a dedicated study, which has to be based on a nonlinear dissipative sloshing model. Further, a special analysis is needed for approximate natural modes, which enable handling singular behaviour at contact line formed by the mean free surface and the rigid walls. Accounting for this singularity should improve convergence.

Nonlinear phenomena are also of importance for resonant sloshing. Results of the present paper may be utilised to improve the multimodal technique (Lukovsky, 1990; Faltinsen et al., 2000; Gavrilyuk et al., 2005) and to study the nonlinear sloshing in a truncated conical tank. This will be the main purpose of our forthcoming studies.
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Table III.

A Vs. $\theta_0$ for $\vartheta_1$.
References


**Appendix**

(The Appendix Figure follows overleaf.)
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**Figure A1.**
Values of $\nu_{mn}$ versus $\theta_0$