A CONSTRUCTION OF GENERALIZED
TRANSLATION OPERATORS

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Abstract. We reconstruct a family of generalized translation operators from the function which generates a given theory of generalization function.

1. Introduction

The biorthogonal approach to a construction of the theory of generalized functions of an infinite number of variables was inspired by [3], proposed in [1] and developed in [1-17] (the paper [11] contains a fairly complete bibliography). The most general results obtained in [4-8], where characters of some family of generalized translation operators were used instead of exponents. Spaces of test function in [4-8] were constructed by its characters.

In [9] the inverse problem is solved in a model one-dimensional case. Namely, for a given function \( h(x, \lambda) \) which generate the theory of generalized functions (this function must satisfy assumptions given in Section 2) it was constructed a family of generalized translation operators for which the function \( h \) is a character.

This article is devoted to solving a corresponding problem in the infinite-dimensional case. We claim that a generalized translation operator is the operator \( h_x(\partial) \) (the so-called annihilation operator of infinite order) associated with the function \( h(x, \lambda) \). Note than such operators were investigated in [13,14] for a special function \( h(x, \lambda) = \gamma(\lambda)\chi((x, \alpha(\lambda))) \), where \( \chi: \mathbb{C} \to \mathbb{C} \) is an entire function, \( \gamma: \mathcal{N}_\mathbb{C} \to \mathbb{C} \) and \( \alpha: \mathcal{N}_\mathbb{C} \to \mathcal{N}_\mathbb{C} \) is a function analytic at \( 0 \in \mathcal{N}_\mathbb{C} \).

2. The spaces of test functions

We use the following notation:

\[ N_p := \{p, p + 1, \ldots\}, \quad p \in \mathbb{Z}, \]

where \( \mathbb{Z} := \{\ldots, -1, 0, 1, \ldots\} \).

Let \( Q \) be a separable complete metric space of points \( x, y, \ldots \). We denote by \( C(Q) \) the linear space of all complex-valued locally bounded (i.e. bounded on

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every ball in $Q$) continuous functions on $Q$. We will understand $C(Q)$ as a linear
topological space with convergence uniform on every ball from $Q$.

For any $p \in \mathbb{N}_1$ we consider a fixed chain of real separable Hilbert spaces,

$$\mathcal{N}_p := \bigcap_{\rho \in \mathbb{N}_1} N_{-\rho} \supset N_{-p} \supset N_0 \supset N_p \supset \lim_{\rho \to \infty} N_{-\rho} =: \mathcal{N},$$

where $N_{-p}$ is the space negative with respect to the positive space $N_p$ and the zero
space $N_0$. We will suppose that the embedding $N_{p+1} \hookrightarrow N_p$, $p \in \mathbb{N}_0$ is quasinuclear
(i.e. the inclusion operator is of the Hilbert-Schmidt type) and, moreover, $\| \cdot \|_{N_p} \leq
\| \cdot \|_{N_{p+1}}$. Let us denote by $\langle \cdot, \cdot \rangle$ the real pairing between
$N_{-p}$ and $N_p$, induced by the scalar product in $N_0$. We will preserve these notations for tensor powers and
complexifications of spaces.

For any $p \in \mathbb{Z}$ and a weight $\gamma = (\gamma_n)_{n=0}^{\infty}$, $\gamma_n > 0$, we can construct a symmetric
weighted Fock space

$$\mathcal{F}(N_p, \gamma) := \bigoplus_{n=0}^{\infty} \mathcal{F}_n(N_p) \gamma_n$$

$$= \left\{ f = (f_n)_{n=0}^{\infty} \mid f_n \in \mathcal{F}_n(N_p), \quad \| f \|_{\mathcal{F}(N_p, \gamma)}^2 = \sum_{n=0}^{\infty} \| f_n \|_{\mathcal{F}_n(N_p)}^2 \gamma_n < \infty \right\},$$

with the corresponding inner product. Here the $n$-particle subspace $\mathcal{F}_n(N_p)$, $p \in \mathbb{Z}$
is equal to the $n$-th symmetric tensor power $\hat{\otimes}$ of the complexification $N_{p, \mathbb{C}}$ of the
space $N_p$, $\mathcal{F}_n(N_p) := N_{p, \mathbb{C}} \hat{\otimes} n$, $N_{p, \mathbb{C}} := \mathbb{C}^1$.

In what follows, we will consider the family $(\mathcal{F}(N_p, \gamma(q)))_{p,q \in \mathbb{N}_1}$, of weighted Fock spaces
$\mathcal{F}(N_p, \gamma(q))$ with the weight

$$\gamma(q) = (\gamma_n(q))_{n=0}^{\infty}, \quad \gamma_n(q) = (n!)^2 K^q, \quad K > 1.$$  

Let $B_0$ be some neighborhood of 0 in the space $N_{1, \mathbb{C}}$ and

$$(2) \quad Q \times B_0 \ni \{ x, \lambda \} \mapsto h(x, \lambda) \in \mathbb{C}^1$$

be a given function. Suppose that for each $x \in Q$ $h(x, \cdot)$ is analytic at $0 \in N_{1, \mathbb{C}}$, and, for each $\lambda \in B_0$, $h(\cdot, \lambda) \in C(Q)$. Moreover, $h(\cdot, \lambda)$ is locally bounded uni-
formly with respect to $\lambda$ from any closed ball inside of $B_0$.

It follows from the analyticity ([11], Subsections 2–3) that, for each point $x \in Q$, there exists a neighborhood of zero

$$B(x) := \{ \lambda \in N_{2, \mathbb{C}} \mid \| \lambda \|_{N_{2, \mathbb{C}}} < R(x), \quad R(x) > 0 \} \subset B_0,$$

such that

$$h(x, \lambda) = \sum_{n=0}^{\infty} \frac{1}{n!} \langle \lambda^n, h_n(x) \rangle, \quad h_n(x) \in \mathcal{F}_n(N_{-2}),$$

for all $\lambda$ from $B(x)$. Moreover, the last series converges uniformly on any closed
ball from $B(x)$. Suppose that for all $x \in Q$ there exists a general neighborhood of
zero

$$B := \{ \lambda \in N_{2, \mathbb{C}} \mid \| \lambda \|_{N_{2, \mathbb{C}}} < R, \quad R > 0 \} \subset B_0$$
with this property.

In accordance with [11] the function
\[ Q \ni x \mapsto \langle f_n, h_n(x) \rangle \in \mathbb{C} \]

belongs to \( C(Q) \) for all \( f_n \in \mathcal{F}_n(N_p), \ n \in \mathbb{N}_0, \ p \in \mathbb{N}_3 \). Moreover ([11], Lemma 4.2), if \( K > 1 \) (here \( K \) from (1)) is sufficiently large, then the series
\[
\sum_{n=0}^{\infty} \langle f_n, h_n(\cdot) \rangle, \quad (f_n)_{n=0}^{\infty} \in \mathcal{F}(N_p, \gamma(q)), \quad p \in \mathbb{N}_3, \ q \in \mathbb{N}_1
\]
converges in the topology of \( C(Q) \) to some function \( f \in C(Q) \).

In what follows, we take \( K > 1 \) sufficiently large. For such fixed \( K > 1 \) and \( p \in \mathbb{N}_3, q \in \mathbb{N}_1 \) we can consider the mapping
\[
(4) \quad \mathcal{F}(N_p, \gamma(q)) \ni f = (f_n)_{n=0}^{\infty} \mapsto (I(p,q)f)(\cdot) := \sum_{n=0}^{\infty} \langle f_n, h_n(\cdot) \rangle \in C(Q).
\]

Suppose that for \( p = 3, \ q = 1 \) the mapping (4) is injective. Then it is obvious that the mapping \( I(p,q) : \mathcal{F}(N_p, \gamma(q)) \to C(Q) \) is injective for any \( p \in \mathbb{N}_3, q \in \mathbb{N}_1 \).

Applying the mapping \( I(p,q) \) we can define the family \( \{H(p,q)\}_{p \in \mathbb{N}_3, q \in \mathbb{N}_1} \) of Hilbert spaces
\[
H(p,q) := I(p,q)(\mathcal{F}(N_p, \gamma(q))]
= \{ f \in C(Q) | \exists (f_n)_{n=0}^{\infty} \in \mathcal{F}(N_p, \gamma(q)) : f(x) = \sum_{n=0}^{\infty} \langle f_n, h_n(x) \rangle, \ x \in Q \}
\]
with the Hilbert norm
\[
\| f \|_{H(p,q)} = \| \sum_{n=0}^{\infty} \langle f_n, h_n(\cdot) \rangle \|_{H(p,q)} := \| (f_n)_{n=0}^{\infty} \|_{\mathcal{F}(N_p, \gamma(q))}.
\]

Remark. We note that the spaces \( H(p,q) \) are the test functions spaces in a generalization of the white noise analysis (see [11] for more details).

3. Annihilation operators

An annihilation operator \( a_-(\xi_m) \) with a coefficient \( \xi_m \in \mathcal{F}_m(N_{-p}), \ m \in \mathbb{N}_0, \) is defined in the Fock space \( \mathcal{F}(N_p, \gamma(q)), \ p \in \mathbb{N}_3, q \in \mathbb{N}_1 \) as linear continuous operator acting by the rule (see [11]): for any \( f = (f_n)_{n=0}^{\infty} \in \mathcal{F}(N_p, \gamma(q)) \)
\[
a_-(\xi_m)f = a_-(\xi_m)(f_0, f_1, \ldots) := (m! f_1^{\xi_m}, \ldots, \frac{n!}{(n-m)!} f_{n-m}^{\xi_m}, \ldots) \in \mathcal{F}(N_p, \gamma(q)),
\]
where \( f_n^{\xi_m} \in \mathcal{F}_{n-m}(N_p), \ n \geq m \) is defined by
\[
\langle f_n, \xi_m \hat{\otimes} \eta_{n-m} \rangle = \langle f_n^{\xi_m}, \eta_{n-m} \rangle
\]
for all \( \eta_{n-m} \in \mathcal{F}_{n-m}(N_{-p}) \).
Using the unitary operator 
\[ \mathcal{F}(N_p, \gamma(q)) \ni f = (f_n)_{n=0}^{\infty} \mapsto (I(p,q)f)(\cdot) = \sum_{n=0}^{\infty} \langle f_n, h_n(\cdot) \rangle \in H(p,q) \]
we transfer the annihilation operator \( a_-(\xi_m) \) into the operator 
\[ \partial(\xi_m) := I(p,q)a_-(\xi_m)I^{-1}(p,q) : H(p,q) \to H(p,q). \]
A simple calculation gives its action on elementary functions \( \langle f_n, h_n(\cdot) \rangle \in H(p,q), \ n \in \mathbb{N}_0 \): for all \( m \in \mathbb{N}_0 \) and \( x \in Q \)
\[ (\partial(\xi_m)(f_n, h_n(\cdot)))(x) := \begin{cases} \frac{n!}{(n-m)!}\langle f_n, \xi_m \hat{\otimes} h_{n-m}(x) \rangle & n \in \mathbb{N}_m; \\ 0 & n = 0, \ldots, m - 1. \end{cases} \] (5)

Let \( \ell : N_{1,\mathbb{C}} \to \mathbb{C}^1 \) be an analytic function at \( 0 \in N_{1,\mathbb{C}} \). Then in some neighborhood of \( 0 \in N_{2,\mathbb{C}} \) there exists an expansion
\[ \ell(\lambda) = \sum_{n=0}^{\infty} \frac{1}{n!} \langle \lambda \hat{\otimes} n, \alpha_n \rangle, \quad \alpha_n \in \mathcal{F}_n(N_{-2}). \]

In accordance with [17] the function \( \ell \) generates a linear continuous operator (the so-called annihilation operator of infinite order)
\[ H(p,q) \ni f \mapsto \ell(\partial)f := \sum_{n=0}^{\infty} \frac{1}{n!} \partial(\alpha_n)f \in H(p,q), \quad p, q \in \mathbb{N}_3. \]
Thus, the function \( h(x,\lambda) \) generates a family \( h(\partial) = (h_x(\partial))_{x \in Q} \) of linear continuous operators
\[ H(p,q) \ni f \mapsto h_x(\partial)f := \sum_{n=0}^{\infty} \frac{1}{n!} \partial(h_n(x))f \in H(p,q), \quad p, q \in \mathbb{N}_3. \]

3. Generalized translation operators

Let a family \( T = (T_x)_{x \in Q} \) of linear operators \( T_x : C(Q) \to C(Q) \) be given. Such a family \( T \) is, by definition (see [8,9,11]), a family of generalized translation operators if
(a) \( (T_x f)(y) = (T_y f)(x) \) for any \( f \in C(Q) \) and \( x, y \in Q \) (commutativity);
(b) there exists a point \( e \in Q \) (basis unity) such that \( T_e = id \);
(c) for any \( x, y \in Q \) the mapping \( C(Q) \ni f \mapsto (T_x f)(y) \in \mathbb{C}^1 \) is continuous (continuity).

Note, that axioms (a)–(c) are only some part of axioms for generalized translation operators from theory of commutative hypercomplex systems and hypergroups, see [10].

Because the embedding \( H(3,3) \hookrightarrow C(Q) \) is continuous (see [11], Theorem 4.1), we can generalize the definition of \( T \). In what follows, we will call \( T = (T_x)_{x \in Q} \) a family of generalized translation operators if the operators \( T_x \) act from the space \( H(3,3) \) into \( C(Q) \) and the following axioms are satisfied:
(a') \( (T_x f)(y) = (T_y f)(x) \) for any \( f \in H(3,3) \) and \( x, y \in Q \) (commutativity);
(b') there exists a point \( e \in Q \) (basis unity) such that \( T_e = id \);
(c') for any \( x, y \in Q \) the mapping \( H(3,3) \ni f \mapsto (T_x f)(y) \in \mathbb{C}^1 \) is continuous (continuity).
We say that a non-zero function \( \chi \in H(3, 3) \) is a character of the family \( T \) if

\[
(T_x \chi)(y) = \chi(x) \chi(y), \quad x, y \in Q.
\]

Without loss of generality one can consider that

\[
h(o, \lambda) = 1,
\]

for some point \( o \in Q \) and all \( \lambda \in V := \{ \lambda \in N_{3, c} \mid ||\lambda||_{N_{3, c}} < r, \ r > 0 \} \) (here \( r > 0 \) sufficiently small). In what follows, we fixed a such point \( o \in Q \).

**Theorem.** The family \( h(\partial) = (h_x(\partial))_{x \in Q} \) of linear continuous operators

\[
h_x(\partial) := \sum_{n=0}^{\infty} \frac{1}{n!} \partial(h_n(x)) : H(3, 3) \to C(Q)
\]

is a family of generalized translation operators. For each fixed \( \lambda \in V \) the function \( Q \ni x \mapsto h(x, \lambda) \in C^1 \) is a character of the family \( h(\partial) \).

If \( h(\cdot, \lambda) \) is a character of some family \( T = (T_x)_{x \in Q} \) of generalized translation operators for all \( \lambda \in V \), then

\[
T_x = h_x(\partial) : H(3, 3) \to C(Q),
\]

for all \( x \in Q \).

**Proof.** Axioms \((a')\), \((b')\), \((c')\) are fulfilled for \( h(\partial) \).

Indeed, since \( h(o, \lambda) = 1 \) for \( \lambda \in V \) we conclude that \( h_o(\partial) = id \) and axiom \((b')\) is fulfilled. The embedding operator \( O : H(3, 3) \hookrightarrow C(Q) \) and operator \( h_3(\partial) : H(3, 3) \to H(3, 3) \) are continuous. Therefore, the operator \( h_x(\partial) : H(3, 3) \to C(Q) \) is continuous and axiom \((c')\) is also fulfilled. The axiom \((a')\) follows from (6) (see below) and axiom \((c')\).

We have to prove that \( h(\cdot, \lambda) \) is a character of family \( (h_x(\partial))_{x \in Q} \) for all \( \lambda \in V \).

Due to (5), the action of the operator \( h_x(\partial) \) on \( \langle f_n, h_n(\cdot) \rangle \in H(3, 3) \), \( n \in \mathbb{N}_0 \) is given by

\[
(h_x(\partial)\langle f_n, h_n(\cdot) \rangle)(y) = \sum_{m=0}^{\infty} \frac{1}{m!} (\partial(h_m(x))\langle f_n, h_n(\cdot) \rangle)(y) = \sum_{m=0}^{n} \frac{n!}{m!(n-m)!} \langle f_n, h_m(x) \hat{\otimes} h_{n-m}(y) \rangle,
\]

(6)

for all \( x, y \in Q \).

The series \( h(\cdot, \lambda) = \sum_{n=0}^{\infty} \frac{1}{n!} (\lambda^{\otimes n} h_n(\cdot)) \), \( \lambda \in V \) converges in the topology of \( H(3, 3) \) ([11], Proposition 4.1) and operator \( h_x(\partial) : H(3, 3) \to C(Q) \) is continuous,
therefore, for any $x, y \in Q$, by (6)

$$(h_x(\partial)h(\cdot, \lambda))(y) = \left( h_x(\partial) \left( \sum_{n=0}^{\infty} \frac{1}{n!} \langle \lambda^{\otimes n}, h_n(\cdot) \rangle \right) \right)(y)$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \langle h_x(\partial)\langle \lambda^{\otimes n}, h_n(\cdot) \rangle \rangle(y)$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \langle \lambda^{\otimes n}, \sum_{m=0}^{n} \frac{n!}{m!(n-m)!} h_m(x) \hat{\otimes} h_{n-m}(y) \rangle$$

$$= \sum_{n=0}^{\infty} \sum_{m=0}^{n} \frac{1}{m!(n-m)!} \langle \lambda^{\otimes m}, h_m(x) \rangle \langle \lambda^{\otimes (n-m)}, h_{n-m}(y) \rangle$$

$$= \left( \sum_{n=0}^{\infty} \frac{1}{n!} \langle \lambda^{\otimes n}, h_n(x) \rangle \right) \left( \sum_{n=0}^{\infty} \frac{1}{n!} \langle \lambda^{\otimes n}, h_n(y) \rangle \right) = h(x, \lambda)h(y, \lambda).$$

Now we prove that if the function $h(\cdot, \lambda)$ is a character of some family $T = (T_x)_{x \in Q}$ of generalized translation operators for all $\lambda \in V$, that

$$T_x = h_x(\partial) : H(3, 3) \to C(Q).$$

for all $x \in Q$.

The mappings

$$(7) \quad H(3, 3) \ni f \to (h_x(\partial)f)(y) \in C^1, \quad H(3, 3) \ni f \to (T_xf)(y) \in C^1$$

are linear and continuous for all $x, y \in Q$. Therefore, it is enough to show that

$$\langle T_x\langle f_n, h_n(\cdot) \rangle \rangle(y) = \langle h_x(\partial)\langle f_n, h_n(\cdot) \rangle \rangle(y)$$

$$= \langle f_n, \sum_{m=0}^{n} \frac{n!}{m!(n-m)!} h_m(x) \hat{\otimes} h_{n-m}(y) \rangle,$$

for any $\langle f_n, h_n(\cdot) \rangle \in H(3, 3)$, $n \in \mathbb{N}_0$ and all $x, y \in Q$.

Fix $x, y \in Q$. It follows from the continuity of the second mapping in (7) that there exists a constant $c > 0$ such that

$$|(T_xf)(y)| \leq c\|f\|_{H(3, 3)}, \quad f \in H(3, 3).$$

Therefore, for $f(\cdot) = \langle f_n, h_n(\cdot) \rangle \in H(3, 3)$, $n \in \mathbb{N}_0$ we have

$$|(T_x\langle f_n, h_n(\cdot) \rangle)(y)| \leq c\|\langle f_n, h_n(\cdot) \rangle\|_{H(3, 3)} = c\|f_n\|_{\mathcal{F}_n(N_3)}.$$

From this estimate we conclude that there exists a unique vector

$$k_n(x, y) \in \mathcal{F}_n(N_{-3})$$

such that

$$\langle T_x\langle f_n, h_n(\cdot) \rangle \rangle(y) = \langle f_n, k_n(x, y) \rangle,$$

for all $f_n \in \mathcal{F}_n(N_3)$. 


Now it is sufficient to prove that

\begin{equation}
\tag{8}
k_n(x, y) = \sum_{m=0}^{n} \frac{n!}{m!(n-m)!} h_m(x) \hat{\otimes} h_{n-m}(y).
\end{equation}

The series \( h(\cdot, \lambda) = \sum_{n=0}^{\infty} \frac{1}{n!} (\lambda^{\otimes n}, h_n(\cdot)) \) converges in the topology of \( H(3,3) \) for all \( \lambda \in V \) and second mapping in (7) is linear and continuous, therefore, for all \( x, y \in Q \) and \( \lambda \in V \) we have

\begin{equation}
\tag{9}
(T_x h(\cdot, \lambda))(y) = \sum_{n=0}^{\infty} \frac{1}{n!} (T_x \lambda^{\otimes n}, h_n(\cdot))(y) = \sum_{n=0}^{\infty} \frac{1}{n!} (\lambda^{\otimes n}, k_n(x, y)).
\end{equation}

On the other hand, according to (3), for all \( x, y \in Q \) and \( \lambda \in V \) we have

\begin{equation}
\tag{10}
(T_x h(\cdot))(y) = h(x, \lambda) h(y, \lambda) = \sum_{n,m=0}^{\infty} \frac{1}{n!m!} (\lambda^{\otimes n}, h_n(x)) (\lambda^{\otimes m}, h_m(y))
= \sum_{n=0}^{\infty} \frac{1}{n!} (\lambda^{\otimes n}, \sum_{m=0}^{n} \frac{n!}{m!(n-m)!} h_m(x) \hat{\otimes} h_{n-m}(y)).
\end{equation}

Let \( z \in \mathbb{C}^1 \) be sufficiently small and \( \varphi \in N_{3; \mathbb{C}}, \| \varphi \|_{N_{3; \mathbb{C}}} = 1 \). By substituting \( \lambda = z \varphi \) in (9), (10) and comparing the coefficients before \( z^n \), we get for \( x, y \in Q \)

\[ \langle \varphi^{\otimes n}, k_n(x, y) \rangle = \langle \varphi^{\otimes n}, \sum_{m=0}^{n} \frac{n!}{m!(n-m)!} h_m(x) \hat{\otimes} h_{n-m}(y) \rangle. \]

The last equality, polarization identity and linearity with respect to \( \varphi^{\otimes n} \) give (8).

**Remark.** It is not difficult to prove that, for all \( x, y, z \in Q \) and \( f \in H(3,3) \), the following relation of associativity holds:

\[ (h_y^x(\partial)(h_y(\partial)f))(x) = (h_y^z(\partial)(h_z(\partial)f))(x), \]

where the notation \( (h_y^z(\partial)(h_y(\partial)f))(x) \) means that the operator \( h_y(\partial) \) acts on the function \( (h_y(\partial)f)(x) \) depending on two variables \( y \) and \( x \) with respect to the variable \( y \).

**Remark.** Let \( T = (T_x)_{x \in Q} \) be a family of generalized translation operators. If \( h(\cdot, \lambda), \lambda \in V \) is a character of the family \( T \), then for each \( p, q \in \mathbb{N}_3 \) the Hilbert space \( H(p, q) \) is invariant with respect to the action of the operator \( T_x \). Moreover, the following equality of operators holds:

\[ T_x = h_x(\partial) : H(p, q) \rightarrow H(p, q). \]

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