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Electro-Static Potential Between Two Conducting Cylinders via the Group Method Approach

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The transformation group theoretic approach is applied to present an analysis of distribution of electro-static potential between two eccentric conducting cylindrical surfaces. A conformal mapping is used to map the region between two eccentric circles, in the complex potential-plane onto a region of concentric circles in another complex-plane. The application of one-parameter group reduces the number of independent variables by one, and consequently the Laplace's equation with the boundary conditions to an ordinary differential equation with appropriate corresponding conditions. The obtained differential equation is solved analytically.

1 Introduction

The Laplace equation arises in many branches of physics, from which it attracts a wide band of researchers. Electro-static potential, temperature in the case of steady state heat conduction, velocity potential in the case of steady irrotational flow of an ideal fluid, concentration of a substance that is diffusing through a solid, and the displacements of a two-dimensional membrane in equilibrium state, are counter examples in which the Laplace's equation is satisfied.

The arrangement of two parallel conducting cylinders, each of circular cross section is an important type of transmission line.

Transmission lines are used to transmit electric energy and signals from one point to another. The basic transmission line connects a source to a load. This may be a transmitter and an antenna, a shift register and the memory core in a digital computer, a hydroelectric generating plant and a substation several hundred miles away, a television antenna and a receiver, and one input of the preamplifier, see Hayt [7]. While short transmission line segments (few millimeters in microwave circuits to inches or feet or hundreds of feet in devices at lower frequencies) perform many different functions, within the terminal units of the systems such as: resonant elements, filters and wave-shaping networks, see Chipman [5].

A conformal mapping is used to map the region between two eccentric circles, in the complex potential-plane onto a region of concentric circles in another complex-plane. Then the mathematical technique used in the present analysis is the parameter-group transformation. The group methods, as a class of methods which lead to reduction of the number of independent variables, were first introduced by Birkhoff [4] in 1948, where he made use of one-parameter transformation groups. In 1952, Morgan [9] presented a theory which has led to improvements over earlier similarity methods. The method has been applied intensively by Abd-el-Malek et al. [1–3].

In this work we present a general procedure for applying one-parameter group transformation to the Laplace's equation in a region between two long cylinders with parallel axis. Under the transformation, the partial differential equation with variable boundary conditions, is reduced to an ordinary differential equation with the appropriate corresponding conditions. The equation is solved analytically.

2 Mathematical formulation

Consider the electro-static potential $V(x, y)$ over any cross section Ω (Fig. 1.1 or Fig. 1.2) of a domain between two long cylinders with parallel axis.

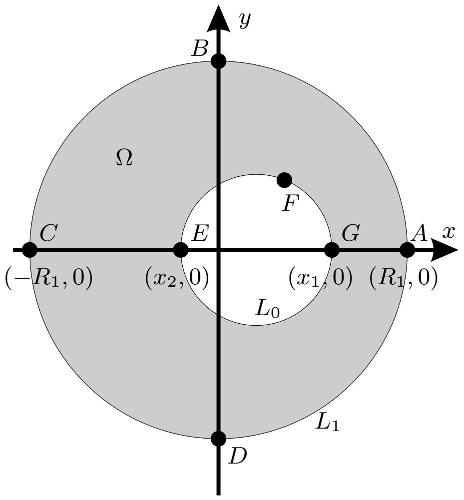

Fig. 1.1. Cross section in two long eccentric cylinders with parallel axis where $-R_1 < x_2 < x_1 < R_1$.

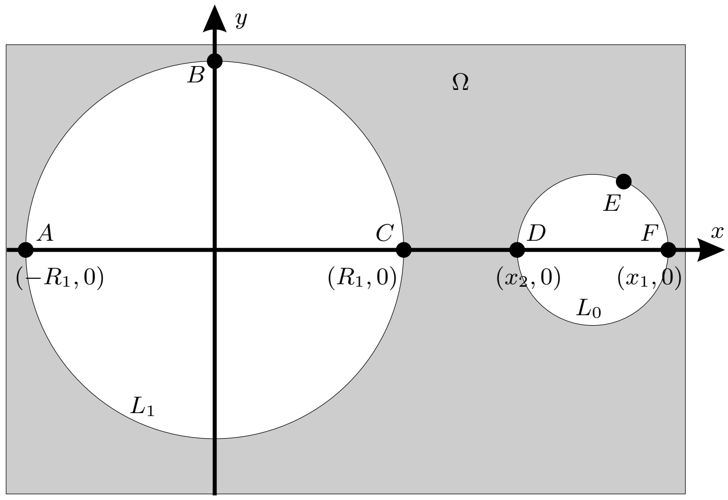

Fig. 1.2. Cross section in two long parallel cylinders where $R_1 < x_2 < x_1$.

Under the assumption that cylinders have variable potentials, the governing equation may be written as

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0, \qquad (x, y) \in \Omega \tag{1}$$

with the boundary conditions

$$\begin{aligned} V(x, y) &= V_1 q(x, y), \qquad (x, y) \in L_1; \\ V(x, y) &= V_0 q(x, y), \qquad (x, y) \in L_0, \end{aligned} \tag{2}$$

where V_1 and V_0 are constants and $q(x, y)$ is an arbitrary function to be determined later on.

The Möbius or the linear fractional transformation

$$w = \frac{z - aR_1}{az - R_1}, \tag{3}$$

where

$$w = u + iv = re^{i\theta}, \qquad z = x + iy,$$

$$a = \frac{R_1^2 + x_1 x_2 + \sqrt{\left(R_2^1 - x_2^1\right)\left(R_2^1 - x_2^2\right)}}{R_1(x_1 + x_2)}, \tag{4}$$

$$R_0 = \frac{R_1^2 - x_1 x_2 + \sqrt{\left(R_1^2 - x_1^2\right)\left(R_1^2 - x_2^2\right)}}{\varepsilon R_1 (x_1 - x_2)}, \tag{5}$$

$\varepsilon = 1$ for Ω shown in Fig. (1.1) and $\varepsilon = -1$ for Ω shown in Fig. (1.2), maps the region Ω shown in Fig. 1.1 onto $\bar{\Omega}$ shown in Fig. 2.1 and also maps the region Ω shown in Fig. 1.2 onto $\bar{\Omega}$ shown in Fig. 2.2, see Churchill and Brown [6].

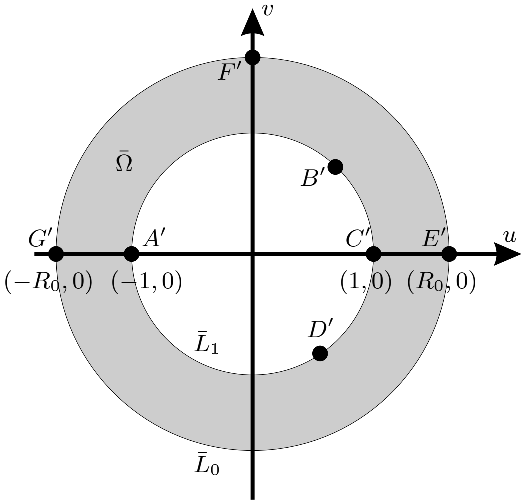

Fig. 2.1. Cross section in mapped two long eccentric cylinders with parallel axis.

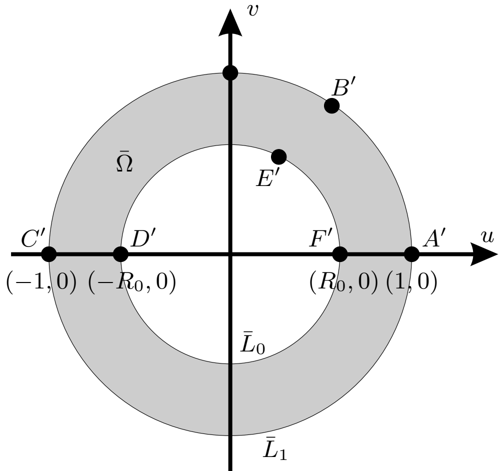

Fig. 2.2. Cross section in mapped two long parallel cylinders.

From (3) we get

$$u = \frac{a\left(x^2+y^2\right) - \left(a^2+1\right) x R_1 + a R_1^2}{a^2\left(x^2+y^2\right) - 2axR_1 + R_1^2}, \tag{6}$$

$$v = \frac{\left(a^2-1\right) y R_1}{a^2\left(x^2+y^2\right) - 2axR_1 + R_1^2}. \tag{7}$$

Now, the governing equation satisfied in region $\bar{\Omega}$, in polar coordinates, has the form

$$\frac{\partial^2 V}{\partial r^2} + \frac{1}{r}\frac{\partial V}{\partial r} + \frac{1}{r^2}\frac{\partial^2 V}{\partial \theta^2} = 0, \tag{8}$$

with the boundary conditions

$$\begin{aligned} &V(1,\theta) = V_1 q(\theta), \qquad (r,\theta) \in \bar{L}_1, \\ &V(R_0,\theta) = V_0 q(\theta), \qquad (r,\theta) \in \bar{L}_0, \end{aligned} \qquad -\pi < \theta \le \pi. \tag{9}$$

We restrict θ to an interval $(-\pi, \pi]$. This requires that:

$$V(r,\pi) = V(r,-\pi), \qquad 1 < r < R_0, \tag{10}$$

$$\frac{\partial V}{\partial \theta}(r,\pi) = \frac{\partial V}{\partial \theta}(r,-\pi), \qquad 1 < r < R_0. \tag{11}$$

Write

$$V(r,\theta) = w(r,\theta) q(\theta), \qquad q(\theta) \not\equiv 0 \quad \text{in} \quad \bar{\Omega}. \tag{12}$$

Hence (8) and (9) take the form:

$$q(\theta)\left[\frac{\partial^2 w}{\partial r^2}+\frac{1}{r}\frac{\partial w}{\partial r}\right]+\frac{1}{r^2}\left[w\frac{d^2 q}{d\theta^2}+2\frac{\partial w}{\partial \theta}\frac{dq}{d\theta}+q\frac{\partial^2 w}{\partial \theta^2}\right]=0 \tag{13}$$

with the boundary conditions:

$$\begin{aligned} w(1,\theta)&=V_1, \qquad (r,\theta)\in \bar{L}_1, \\ w(R_0,\theta)&=V_0, \qquad (r,\theta)\in \bar{L}_0, \end{aligned} \qquad -\pi<\theta<\pi. \tag{14}$$

3 Solution of the problem

The method of solution depends on the application of a one-parameter group transformation to the partial differential equation (13) and the boundary conditions (14). Under this transformation the two independent variables will be reduced by one and the differential equation (13) transforms into an ordinary differential equation in only one independent variable, which is the similarity variable.

3.1 The group systematic formulation

The procedure is initiated with the group G, a class of transformation of one-parameter “b” of the form

$$G:\bar{S}=C^S(b)S+K^S(b), \tag{15}$$

where S stands for r, θ; w, q and the C^S and K^S are real-valued and at least differentiable in the real argument “b”.

3.2 The invariance analysis

To transform the differential equation, transformations of the derivatives are obtained from G via chain-rule operations:

$$\bar{S}_{\bar{i}}=\left(\frac{C^S}{C^i}\right)S_i, \qquad \bar{S}_{\bar{i}\bar{j}}=\left(\frac{C^S}{C^iC^j}\right)S_{ij}, \qquad i=r,\theta; \quad j=r,\theta, \tag{16}$$

where S stands for w and q.

Equation (13) is said to be invariantely transformed whenever

$$\begin{aligned} &\bar{q}\left(\bar{r}^2\bar{w}_{\bar{r}\bar{r}}+\bar{r}\bar{w}_{\bar{r}}\right)+\bar{w}\bar{q}_{\bar{\theta}\bar{\theta}}+2\bar{w}_{\bar{\theta}}\bar{q}\bar{w}_{\bar{\theta}\bar{\theta}} \\ &\quad=H(b)\left[q\left(r^2w_{rr}+rw_r\right)+wq_{\theta\theta}+2w_\theta q_\theta+qw_{\theta\theta}\right] \end{aligned} \tag{17}$$

for some function $H(b)$ which may be a constant.

Substitution from equations (15) into equation (17), using (16), for the independent variables, the functions and their partial derivatives yields

$$\begin{aligned} &q\left(\left[C^qC^w\right]r^2w_{rr}+\left[C^qC^w\right]rw_r\right)+\left[\frac{C^qC^w}{\left(C^\theta\right)^2}\right]wq_{\theta\theta}+2\left[\frac{C^qC^w}{\left(C^\theta\right)^2}\right]w_\theta q_\theta \\ &\quad+\left[\frac{C^qC^w}{\left(C^\theta\right)^2}\right]qw_{\theta\theta}+\zeta(b)=H(b)\left[q\left(r^2w_{rr}+rw_r\right)+wq_{\theta\theta}+2w_\theta q_\theta+wq_{\theta\theta}\right], \end{aligned} \tag{18}$$

where

$$\zeta(b) = K^q \left[(C^r r + K^r)^2 \left(\frac{C^w}{(C^r)^2} \right) w_{rr} + (C^r r + K^r) \left(\frac{C^w}{C^r} \right) w_r \right] + \left[\frac{K^w C^q}{(C^\theta)^2} \right] q_{\theta\theta} + K^q \left[\frac{C^w}{(C^\theta)^2} \right] w_{\theta\theta}. \tag{19}$$

The invariance of (18) implies $\zeta(b) \equiv 0$. This is satisfied by putting

$$K^q = K^w = 0 \tag{20}$$

and

$$[C^q C^w] = \left[\frac{C^q C^w}{(C^\theta)^2} \right] = H(b), \tag{21}$$

which yields

$$C^\theta = \pm 1. \tag{22}$$

Moreover, the boundary conditions (14) are also invariant in form, imply that $K^r = K^w = 0$ and $C^w = C^r = 1$.

Finally, we get the one-parameter group G which transforms invariantely the differential equation (13) and the boundary conditions (14). The group G is of the form

$$G : \quad \begin{cases} \bar{r} = r, \\ \bar{\theta} = \pm\theta + K^\theta, \\ \bar{w} = w, \\ \bar{q} = C^q q. \end{cases} \tag{23}$$

3.3 The complete set of absolute invariant

Our aim is to make use of group methods to represent the problem in the form of an ordinary differential equation (similarity representation) in a single independent variable (similarity variable). Then we have to proceed in our analysis to obtain a complete set of absolute invariant. In addition to the absolute invariant of the independent variable, there are two absolute invariant of the dependent variables w and q.

If $\eta \equiv \eta(r, \theta)$ is an absolute invariant of the independent variables, then

$$g_j(r, \theta; w, q) = F_j[\eta, (r, \theta)], \qquad j = 1, 2 \tag{24}$$

are two absolute invariant corresponding to w and q. The application of a basic theorem in group theory, see Morgan and Gaggioli [8], states that: *a function $g(r, \theta; w, q)$ is an absolute invariant of a one-parameter group if it satisfies the following first-order linear differential equation*

$$\sum_{i=1}^{4} (\alpha_i S_i + \beta_i) \frac{\partial g}{\partial S_i} = 0, \tag{25}$$

where S_i, stands for r, θ, w and q, respectively and

$$\alpha_i = \frac{\partial C^{S_i}}{\partial b} \left(b^0 \right) \qquad \text{and} \qquad \beta_i = \frac{\partial K^{S_i}}{\partial b} \left(b^0 \right), \qquad i = 1, 2, 3, 4 \tag{26}$$

where b^0 denotes the value of "b" which yields the identity element of the group.

From which we get: $\alpha_1 = \alpha_2 = \alpha_3 = 0$ and $\beta_1 = \beta_3 = \beta_4 = 0$.

At first, we seek an absolute invariant of the independent variables. Owing to equation (11), $\eta(r, \theta)$ is an absolute invariant if it satisfies the first-order partial differential equation

$$\frac{\partial \eta}{\partial \theta} = 0, \tag{27}$$

which has a solution in the form $\eta(r, \theta) = \Gamma(r)$.

Without loss of generality, take the function Γ as an identity function, hence

$$\eta(r, \theta) = r. \tag{28}$$

The second step is to obtain the absolute invariant of the dependent variables w and q. Applying (25), we get

$$q(\theta) = R(\theta)\Phi(\eta). \tag{29}$$

Since $q(\theta)$ and $R(\theta)$ are independent of η, while Φ is a function of η, then $\phi(\eta)$ must be a constant, say $\Phi(\eta) = 1$, and from which

$$q(\theta) = R(\theta), \tag{30}$$

and the second absolute invariant is:

$$w(r, \theta) = F(\eta). \tag{31}$$

From (30), (31), and (12). the conditions (10) and (11) will be changed to corresponding conditions on $R(\theta)$ as follows

$$R(\pi) = R(-\pi), \tag{32}$$

$$\frac{dR}{d\theta}(\pi) = \frac{dR}{d\theta}(-\pi). \tag{33}$$

4 Reduction to ordinary differential equation

As the general analysis proceeds, the established forms of the dependent and independent absolute invariant are used to obtain ordinary differential equation. Generally, the absolute invariant $\eta(r, \theta)$ has the form given in (28).

Substituting from (28), (30) and (31) into equation (13) yields

$$\frac{d^2F}{d\eta^2} + \frac{1}{\eta}\frac{dF}{d\eta} + \left(\frac{1}{R\eta^2}\frac{d^2R}{d\theta^2}\right) F = 0. \tag{34}$$

For (34) to be reduced to an expression in the single independent invariant η, the coefficients in (34) should be constants or functions of η. Thus take

$$\frac{1}{R}\frac{d^2R}{d\theta^2} = C, \tag{35}$$

where C is an arbitrary constant.

Thus (34) may be written as

$$\eta^2\frac{d^2F}{d\eta^2} + \eta\frac{dF}{d\eta} + CF = 0. \tag{36}$$

Under the similarity variable η, the boundary conditions are:

$$\begin{aligned} F(R_0) &= V_0, \\ F(1) &= V_1. \end{aligned} \tag{37}$$

5 Analytical solution

Case (1): $C = -\alpha^2$, $\alpha \neq 0$. Substituting for C into (35), we get

$$\frac{d^2R}{d\theta^2} + \alpha^2 R = 0. \tag{38}$$

Solution of (38) is:

$$R(\theta) = c_1 \cos\alpha\theta + c_2 \sin\alpha\theta. \tag{39}$$

The function $R(\theta)$ in (39) satisfies boundary conditions (32) and (33) for $\alpha = n$, where $n = \pm1, \pm2, \pm3, \ldots$.

Since $q(\theta)$ is an arbitrary function, then $R(\theta)$ is also an arbitrary function. Thus we can take R(9) in the form

$$R(\theta) = \sin\theta, \qquad -\pi < \theta \leq \pi, \tag{40}$$

and consequently $q(x, y)$ has the form

$$q(x,y) = \sin\left[\tan^{-1}\left(\frac{\left(a^2-1\right) yR_1}{a\left(x^2+y^2\right) - \left(a^2+1\right) xR_1 + aR_1^2}\right)\right].$$

Substituting for C into (36), we get

$$\eta^2 \frac{d^2F}{d\eta^2} + \eta\frac{dF}{d\eta} - n^2 F = 0, \tag{41}$$

which has the solution

$$F(\eta) = k_1\eta^n + \frac{k_2}{\eta^n}, \tag{42}$$

where k_1 and k_2 are constant.

Applying boundary conditions (37), we get

$$F(\eta) = \frac{V_1 - V_0R_0^n}{1 - R_0^{2n}}\eta^n + \frac{V_0R_0^n - V_1R_0^{2n}}{\left(1 - R_0^{2n}\right)\eta^n}, \qquad n = \pm1, \pm2, \pm3, \ldots. \tag{43}$$

From which we get

$$V(x,y) = \left(\frac{\sin\theta}{1-R_0^{2n}}\right)\left[\left(V_1 - V_0R_0^n\right)\sqrt{u^2+v^2} + \frac{V_0R_0^n - V_1R_0^{2n}}{\sqrt{u^2+v^2}}\right], \quad n = \pm1, \pm2, \pm3, \ldots, \tag{44}$$

where $\theta = \tan^{-1}(v/u)$, u and v are given by (6) and (7), respectively.

To obtain the electrostatic potential in the case of coaxial cylinders, take limit as $a \to \infty$ in (44), and for $n = 1$, we get

$$V(\rho,\Psi) = \frac{\sin\Psi}{\rho_1^2 - \rho_0^2}\left[\left(V_1\rho_0^2 - V_0\rho_0\rho_1\right)\left(\frac{\rho_1}{\rho}\right) + \left(V_0\rho_0 - V_1\rho_1\right)\rho\right], \tag{45}$$

where ρ_0 and ρ_1 are the raduii of the inner and outer cylinders, respectively.

It is noticed that the case of $c = \alpha^2$ is neglected since the obtained form of the function $R(\theta)$ does not satisfy the conditions (32) and (33).

Case (2): $C = 0$. Substituting for C into (35), we get

$$\frac{d^2R}{d\theta^2} = 0. \tag{46}$$

Equation (46) has the solution

$$R(\theta) = c_3\theta + c_4. \tag{47}$$

Since $q(\theta)$ is an arbitrary function, then $R(\theta)$ is also an arbitrary function. Thus we can take, without loss of generality, $R(\theta)$ in the form

$$R(\theta) = 1, \tag{48}$$

which satisfies conditions (32) and (33).

This corresponds to $q(x, y) = 1$, i.e. constant potential along the boundary.

Substituting for C into (36), we get

$$\eta^2\frac{d^2F}{d\eta^2} + \eta\frac{dF}{d\eta} = 0. \tag{49}$$

Equation (49) has the solution

$$F(\eta) = k_3 + k_4 \ln\eta, \tag{50}$$

where k_3 and k_4 are constants.

With the aid of the boundary conditions (37), the solution is

$$F(\eta) = V_1 + \left(\frac{V_0 - V_1}{\ln R_0}\right)\ln\eta, \tag{51}$$

from which we get

$$V(x, y) = V_1 + \left(\frac{V_0 - V_1}{2\ln R_0}\right)\ln\left(u^2 + v^2\right), \tag{52}$$

where u and v are given by (6) and (7) repectively.

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