

Equivalence of Q -Conditional Symmetries under Group of Local Transformation

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The definition of Q -conditional symmetry for one PDE is correctly generalized to a special case of systems of PDEs and involutive families of operators. The notion of equivalence of Q -conditional symmetries under a group of local transformation is introduced. Using this notion, all possible single Q -conditional symmetry operators are classified for the n -dimensional ($n \geq 2$) linear heat equation and for the Euler equations describing the motion of an incompressible ideal fluid.

The concept of Q -conditional symmetry called also nonclassical symmetry was introduced by Bluman and Cole in 1969. This year is the year of the 30th anniversary of appearance of their pioneering paper [1]. Although the concept of Q -conditional symmetry exists for a long time and has various applications many problem of its theory are not solved so far.

Before 1986 the nonclassical symmetry was only mentioned in few papers, e.g., in [2]. The intensive application of Q -conditional symmetries to finding exact solutions of partial differential equations (PDEs) and the parallel search for their foundations was begun after publication of the papers of Olver and Rosenau in 1986 and 1987 [3, 4] as well as the paper of Fushchych and Tsyfra in 1987 [5].

The first correct definition of a Q -conditional symmetry operator for one PDE was given in [5]. Later it was generalized to involutive families of operators [6–8]. We stress that it can be directly extended only to some special cases of Q -conditional invariance for systems of PDEs. For all the other cases this definition must be essentially modified and is much more complicated.

In this paper we correctly generalize the definition of Q -conditional symmetry [6–8] to a special case of systems of PDEs and involutive families of operators. Further, we introduce the notion of equivalence of Q -conditional symmetries under a group of local transformation. Using this notion, we can, first, classify all the possible Q -conditional symmetries and, correspondingly, all the possible reductions of systems of PDEs [7, 8] and, secondly, essentially simplify the procedure of finding Q -conditional symmetries in some cases when the Lie symmetry group is sufficiently wide.

Consider a system of k PDEs of the order r for m unknown functions $u = (u^1, \dots, u^m)$ depending on n independent variables $x = (x_1, \dots, x_n)$ of the form

$$L(x, u_{(r)}(x)) = 0, \quad L = (L^1, \dots, L^k). \quad (1)$$

Here the order of a system is the order of the major partial derivative appearing in the system. The symbol $u_{(r)}$ denotes for the set of partial derivatives of the functions u of the orders from 0 to r . Within the local approach system (1) is treated as a system of algebraic equations in the jet space $J^{(r)}$ of the order r .

Consider also an involutive family Q of l differential operators

$$Q^s = \xi^{si}(x, u)\partial_{x_i} + \eta^{sa}(x, u)\partial_{u_a}, \quad \text{where} \quad l \leq n, \quad \text{rank} \|\xi^{si}(x, u)\| = l. \quad (2)$$

The requirement of involution means for the family Q that the commutator of any pair of operators from Q belongs to the span of Q over the ring of smooth functions of the variables x and u , i.e.

$$\forall s, p \quad \exists \zeta^{sps'} = \zeta^{sps'}(x, u): \quad [Q^s, Q^p] = \zeta^{sps'} Q^{s'}. \tag{3}$$

Here and below the indices a and b run from 1 to m , the indices i and j run from 1 to n , the indices s and p run from 1 to l , and the indices μ and ν run from 1 to $n - l$. The summation is imposed over the repeated indices. Subscripts of functions denote differentiation with respect to the corresponding variables.

If operators (2) form an involutive family, then the family \tilde{Q} of differential operators

$$\tilde{Q}^s = \lambda^{sp} Q^p, \quad \text{where} \quad \lambda^{sp} = \lambda^{sp}(x, u), \quad \det \|\lambda^{sp}\| \neq 0, \tag{4}$$

is also involutive. And family (4) is called *equivalent* to family (2) [6–8].

Notation: $\tilde{Q} = \{\tilde{Q}^s\} \sim Q = \{Q^s\}$.

By the Frobenius theorem, condition (3) is sufficient for the system of PDEs

$$Q^s[u^a] := \eta^{sa}(x, u) - \xi^{si}(x, u) \frac{\partial u^a}{\partial x_i} = 0 \tag{5}$$

to be compatible.

Denote the manifold defined by the system of algebraic equations $L = 0$ in $J^{(r)}$ by \mathcal{L} and the manifold corresponding to the set of all the differential consequences of the system of PDEs (5) in $J^{(r)}$ by \mathcal{M} :

$$\begin{aligned} \mathcal{L} &= \{(x, u_{(r)}) \in J^{(r)} \mid L(x, u_{(r)}) = 0\}, \\ \mathcal{M} &= \{(x, u_{(r)}) \in J^{(r)} \mid D_1^{\alpha_1} \dots D_n^{\alpha_n} Q^s[u^a] = 0, \alpha_i \in \mathbb{N} \cup \{0\}, |\alpha| := \alpha_1 + \dots + \alpha_n < r\}, \end{aligned}$$

where $D_i = \partial_{x_i} + \sum_{\alpha} u_{\alpha,i}^a \partial_{u_{\alpha}^a}$ is the operator of total differentiation with respect to the variable x_i , u_{α}^a and $u_{\alpha,i}^a$ denote the variables in $J^{(r)}$, corresponding to derivatives $\frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}}$ and $\frac{\partial^{|\alpha|+1} u}{\partial x_1^{\alpha_1} \dots \partial x_{i-1}^{\alpha_{i-1}} \partial x_i^{\alpha_i+1} \partial x_{i+1}^{\alpha_{i+1}} \dots \partial x_n^{\alpha_n}}$.

Let the system $L|_{\mathcal{M}} = 0$ do not includes equations which are differential consequences of other its equations. Moreover, let all the differential consequences of the system $L|_{\mathcal{M}} = 0$, the orders of which (as equations) are less than or equal to its order, vanish on $\mathcal{L} \cap \mathcal{M}$.

Definition 1. *System of smaller PDEs (1) is called Q -conditional invariant with respect to involutive family of differential operators (2) if the relation*

$$\left(Q_{(r)}^s L \right) \Big|_{\mathcal{M} \cap \mathcal{L}} = 0 \tag{6}$$

holds true. Here the symbol $Q_{(r)}^s$ denotes the r th prolongation of the operator Q^s :

$$Q_{(r)}^s = Q^s + \sum_{|\alpha| \leq r} \eta^{sa\alpha} \partial_{u_{\alpha}^a}, \quad \eta^{sa\alpha} = D_1^{\alpha_1} \dots D_n^{\alpha_n} (\eta^{sa} - \xi^{si} u_i^a) + \xi^{si} u_{\alpha,i}^a.$$

Denote the set of involutive families of l operators of Q -conditional symmetry of system (1) as $\mathcal{B}(\mathcal{L}, l)$:

$$\mathcal{B}(\mathcal{L}, l) = \left\{ Q = \{Q^1, \dots, Q^l\} \mid \begin{array}{l} \text{the system } L = 0 \text{ is } Q\text{-conditionally} \\ \text{invariant with respect to } Q \end{array} \right\}$$

Lemma [6–8]. *Let system of PDEs (1) be Q -conditionally invariant with respect to involutive family of operators (2). Then, it is Q -conditionally invariant with respect to an arbitrary family of the form (4), i.e.*

$$Q \in \mathcal{B}(\mathcal{L}, l), \tilde{Q} \sim Q \implies \tilde{Q} \in \mathcal{B}(\mathcal{L}, l).$$

An important consequence of the lemma is that we can study Q -conditionally invariance up to equivalence relation (4) which is defined on the set of involutive families of l operators as well as in $\mathcal{B}(\mathcal{L}, l)$. Then it is possible for an arbitrary family of operators (2) to choose the functions $\lambda^{sp}(x, u)$ and, if it is necessary, to change enumeration of the variables x_1, \dots, x_n in such a way that operators (4) take the following form: $\hat{Q}^s = \partial_{x_s} + \hat{\xi}^{s,l+\nu} \partial_{x_{l+\nu}} + \hat{\eta}^{sa} \partial_{u^a}$. Operators \hat{Q}^s generate a commutative Lie algebra.

Let $A(\mathcal{L})$ and $G(\mathcal{L})$ denote the maximal Lie invariance algebra of system (1) and its maximal local symmetry correspondingly. Now we strengthen the equivalence relation in $\mathcal{B}(\mathcal{L}, l)$, given by formula (4), by means of generalizing equivalence of l -dimensional subalgebras of the algebra $A(\mathcal{L})$ under the adjoint representation of the group $G(\mathcal{L})$ in $A(\mathcal{L})$.

We use the following lemma for this generalization.

Lemma. *Let g be an arbitrary local transformation from $G(\mathcal{L})$. Then the adjoint action of g in the set of differential operators generate a one-to-one mapping from $\mathcal{B}(\mathcal{L}, l)$ into itself.*

Let $Q = \{Q^s\}$ and $\tilde{Q} = \{\tilde{Q}^s\}$ be involutive families of differential operators.

Definition. *The families Q and \tilde{Q} are called equivalent with respect to a group G of local transformations if there exists a local transformation g from G for which the families Q and $\text{Ad}(g)\tilde{Q}$ are equivalent.*

Notation: $Q \sim \tilde{Q} \text{ mod } G$.

Definition. *The families Q and \tilde{Q} are called equivalent with respect to a Lie algebra A of differential operators if they are equivalent with respect to the one-parametric group generated by an operator from A .*

Notation: $Q \sim \tilde{Q} \text{ mod } A$.

Therefore,

$$Q \sim \tilde{Q} \text{ mod } G \stackrel{\text{def}}{\iff} \exists g \in G: Q \sim \text{Ad}(g)\tilde{Q}. \tag{7}$$

$$Q \sim \tilde{Q} \text{ mod } A \stackrel{\text{def}}{\iff} \exists V \in A: Q \sim \tilde{Q} \text{ mod } \{e^{\varepsilon V}, \varepsilon \in U(0, \delta) \subset \mathbb{R}\} \tag{8}$$

Lemma. *Formulas (7) and (8) define equivalence relations in the set of involutive families of l differential operators. Moreover, if G is a subgroup of $G(\mathcal{L})$ and A is a subalgebra of $A(\mathcal{L})$ then formulas (7) and (8) define equivalence relations in $\mathcal{B}(\mathcal{L}, l)$.*

Consider two examples.

Example 1. Investigate Q -conditional invariance of the linear n -dimensional heat equation

$$u_t = u_{aa}, \quad \text{where } u = u(t, \vec{x}), \quad t = x_0, \quad \vec{x} = (x_1, \dots, x_n), \tag{9}$$

with respect to a single operator ($l = 1$).

It is just the problem with $n = 1$ for which Bluman and Cole introduced the concept of nonclassical symmetry. In the one-dimensional case the problem was completely solved in [9]. That is why we pay our attention to the multidimensional problem.

Lie symmetry of equation (9) is well known. In the one-dimensional case it was investigated by Lie. The maximal Lie invariance algebra $A(\text{LHE})$ of (9) is generated by the following operators:

$$\begin{aligned} \partial_t &= \partial/\partial t, & \partial_a &= \partial/\partial x_a, & D &= 2t\partial_t + x_a\partial_a, & G_a &= t\partial_a - \frac{1}{2}x_a u\partial_u, & I &= u\partial_u, \\ J_{ab} &= x_a\partial_b - x_b\partial_a \quad (a < b), & \Pi &= 4t^2 + 4tx_a\partial_a - (x_a x_a + 2t)u\partial_u, & f(t, \vec{x}) &= f(t, \vec{x})\partial_u, \end{aligned} \quad (10)$$

where $f = f(t, \vec{x})$ is an arbitrary solution of (9).

Theorem 1. *For any operator Q of Q -conditional symmetry of equation (9) one of three following conditions holds:*

1. $Q \sim \tilde{Q}^0$, where $\tilde{Q}^0 \in A(\text{LHE})$;
2. $Q \sim \tilde{Q}^1 = \partial_n + g_n g^{-1} u \partial_u \bmod \text{ASO}(n) + A^\infty(\text{LHE})$, where $g = g(t, x_n)$ ($g_n \neq 0$) is a solution of the one-dimensional heat equation, that is, $g_t = g_{nn}$;
3. $Q \sim \tilde{Q}^2 = J_{12} + \varphi(\theta)u\partial_u \bmod \text{AG}(1, n) + A^\infty(\text{LHE})$, where $\varphi = \varphi(\theta)$ is a solution of the equation $\varphi_{\theta\theta} + 2\varphi\varphi_\theta = 0$, $\varphi_\theta \neq 0$, θ is the polar angle in the plane OX_1X_2 .

Here $A^\infty(\text{LHE}) = \langle f(t, \vec{x})\partial_u | f = f(t, \vec{x}): f_t = f_{aa} \rangle$, $\text{AG}(1, n) = \langle \partial_t, \partial_a, G_a, J_{ab} \rangle$, $\text{ASO}(n) = \langle J_{ab} \rangle$.

It follows from Theorem 1 that there exist only three classes of the possible reductions on one independent variable for the linear multidimensional heat equation.

The first class is formed by Lie reductions.

The second class involves reductions which are similar to separation of variables in the Cartesian coordinates:

$$u = g(t, x_n)v(\omega_0, \dots, \omega_{n-1}), \quad \text{where } \omega_0 = t, \omega_i = x_i; \quad (9) \implies v_0 = v_{ii}.$$

The third class is formed by reductions which are similar to separation of variables in the cylindrical coordinates:

$$\begin{aligned} u &= \exp\left(\int \varphi(\theta)d\theta\right)v(\omega_0, \dots, \omega_{n-1}), \quad \text{where } \omega_0 = t, \omega_1 = r, \omega_s = x_{s+1}, s = \overline{2, n-1}; \\ (9) &\implies v_0 = v_{11} + \omega_1^{-1}v_1 - \lambda\omega_1^{-2}v + v_{ss}. \end{aligned}$$

Here $\lambda = -\varphi_\theta - \varphi^2 = \text{const}$, (r, θ) are the polar coordinates in the plane OX_1X_2 . As the equation $\varphi_{\theta\theta} + 2\varphi\varphi_\theta = 0$ has four essentially different (under translations with respect to θ) families of solutions with $\varphi_\theta \neq 0$, there are four inequivalent cases for the third class of reductions ($\varkappa \neq 0$):

- a) $\varphi = -\varkappa \tan \varkappa\theta$: $u = v(\omega_0, \dots, \omega_{n-1}) \cos \varkappa\theta$, $\lambda = \varkappa^2$;
- b) $\varphi = \varkappa \tanh \varkappa\theta$: $u = v(\omega_0, \dots, \omega_{n-1}) \cosh \varkappa\theta$, $\lambda = -\varkappa^2$;
- c) $\varphi = \varkappa \coth \varkappa\theta$: $u = v(\omega_0, \dots, \omega_{n-1}) \sinh \varkappa\theta$, $\lambda = -\varkappa^2$;
- d) $\varphi = \theta^{-1}$: $u = v(\omega_0, \dots, \omega_{n-1})\theta$, $\lambda = 0$.

Example 2. Consider the Euler equations

$$\vec{u}_t + (\vec{u} \cdot \nabla)\vec{u} + \nabla p = \vec{0}, \quad \text{div } \vec{u} = 0 \quad (11)$$

describing the motion of an incompressible ideal fluid. In the following $\vec{u} = \{u^a(t, \vec{x})\}$ denotes the velocity of the fluid, $p = p(t, \vec{x})$ denotes the pressure, $n = 3$, $\vec{x} = \{x_a\}$, $\partial_t = \partial/\partial t$, $\partial_a = \partial/\partial x_a$, $\vec{\nabla} = \{\partial_a\}$, $\Delta = \vec{\nabla} \cdot \vec{\nabla}$ is the Laplacian. The fluid density is set equal to unity.

Lie symmetry of system (11) was investigated by Buchnev [10, 11]. The maximal Lie invariance algebra $A(E)$ of (11) is infinite dimensional and generated by the following operators:

$$\begin{aligned} \partial_t, \quad J_{ab} &= x_a \partial_b - x_b \partial_a + u^a \partial_{u^b} - u^b \partial_{u^a} \quad (a < b), \\ D^t &= t \partial_t - u^a \partial_{u^a} - 2p \partial_p, \quad D^x = x_a \partial_a + u^a \partial_{u^a} + 2p \partial_p, \\ R(\vec{m}) &= m^a(t) \partial_a + m_t^a(t) \partial_{u^a} - m_{tt}^a(t) x_a \partial_p, \quad Z(\chi) = \chi(t) \partial_p, \end{aligned} \quad (12)$$

where $m^a = m^a(t)$ and $\chi = \chi(t)$ are arbitrary smooth functions of t (for example, from $C^\infty((t_0, t_1), \mathbb{R})$). Let us investigate Q -conditional symmetry of (11) with respect to alone differential operator $Q = \xi^0(t, \vec{x}, \vec{u}, p) \partial_t + \xi^a(t, \vec{x}, \vec{u}, p) \partial_a + \eta^a(t, \vec{x}, \vec{u}, p) \partial_{u^a} + \eta^0(t, \vec{x}, \vec{u}, p) \partial_p$.

Theorem 2. *Any operator Q of Q -conditional symmetry of the Euler equations (11) either is equivalent to a Lie symmetry operator of (11) or is equivalent (mod $A(E)$) to the operator*

$$\tilde{Q} = \partial_3 + \zeta(t, x_3, u^3) \partial_{u^3} + \chi(t) x_3 \partial_p, \quad (13)$$

where $\zeta_{u^3} \neq 0$, $\zeta_3 + \zeta \zeta_{u^3} = 0$, $\zeta_t + (u^3 \zeta + \chi x_3) \zeta_{u^3} + (\zeta)^2 + \chi = 0$.

It follows from Theorem 2 that there exist two classes of the possible reductions w.r.t. independent variable for the Euler equations, namely, the Lie reductions and the reductions corresponding to operators of form (13).

Lie reductions of the Euler equations (11) are investigated in [12–14].

An ansatz constructed with the operator \tilde{Q} has the following form:

$$u^1 = v^1, \quad u^2 = v^2, \quad u^3 = x_3 v^3 + \psi(t, v^3), \quad p = q + \frac{1}{2} \chi(t) x_3^2,$$

where $v^a = v^a(t, x_1, x_2)$, $q = q(t, x_1, x_2)$, $\psi = \psi(t, v^3)$ is a solution of the equation

$$\psi_t - \left((v^3)^2 + \chi \right) \psi_{v^3} + v^3 \psi = 0.$$

Substituting this ansatz into (11), we obtain the corresponding reduced system ($i, j = 1, 2$):

$$v_t^i + v^j v_j^i + q_i = 0, \quad v_t^3 + v^j v_j^3 + (v^3)^2 + \chi = 0, \quad v_j^j + v^3 = 0.$$

The analogous problem for the Navier–Stokes equations

$$\vec{u}_t + (\vec{u} \cdot \nabla) \vec{u} + \nabla p - \nu \Delta \vec{u} = \vec{0}, \quad \operatorname{div} \vec{u} = 0 \quad (\nu \neq 0) \quad (14)$$

describing the motion of an incompressible viscous fluid was solved by Ludlow, Clarkson, and Bassom in [15]. Their result can be reformulated as follows.

Theorem 3. *Any (real) operator Q of Q -conditional symmetry of the Navier–Stokes equations (14) is equivalent to a Lie symmetry operator of (14).*

Therefore, all the possible reductions of the Navier–Stokes equations w.r.t. independent variable are exhausted by the Lie reductions. Lie symmetry of system (14) was studied by Danilov [16, 17]. The maximal Lie invariance algebra of the Navier–Stokes equations (14) is similar to one of the Euler equations (see (12)):

$$A(NS) = \langle \partial_t, J_{ab}, D^t + \frac{1}{2} D^x, R(\vec{m}(t)), Z(\zeta(t)) \rangle.$$

The Lie reductions of the Navier–Stokes equations were completely described in [18].

References

- [1] Bluman G. and Cole J.D., The general similarity solution of the heat equation, *J. Math. Mech.*, 1969, V.18, N 11, 1025–1042.
- [2] Harrison B.K. and Estabrook F.B., Geometric Approach to Invariance Groups and Solutions of Partial Differential Systems, *J. Math. Phys.*, 1969, V.12, N 4, 653–666.
- [3] Olver P.J. and Rosenau P., The construction of special solutions to partial differential equations, *Phys. Lett. A*, 1986, V.114, N 3, 107–112.
- [4] Olver P.J. and Rosenau P., Group-invariant solutions of differential equations, *SIAM J. Appl. Math.*, 1987, V.47, N 2, 263–278.
- [5] Fushchych W.I. and Tsyfra I.M., On a reduction and solutions of nonlinear wave equations with broken symmetry, *J. Phys. A: Math. Gen.*, 1987, V.20, N 2, L45–L48.
- [6] Fushchych W.I. and Zhdanov R.Z., Conditional symmetry and reduction of partial differential equations, *Ukr. Math. J.*, 1992, V.44, N 7, 970–982.
- [7] Zhdanov R.Z. and Tsyfra I.M., Reduction of differential equations and conditional symmetry, *Ukr. Math. J.*, 1996, V.48, N 6, 595–602.
- [8] Zhdanov R.Z., Tsyfra I.M. and Popovych R.O., A precise definition of reduction of partial differential equations, *J. Math. Anal. Appl.*, 1999, V.238, N 1, 101–123.
- [9] Fushchych W.I., Shtelen W.M., Serov M.I. and Popovych R.O., Q -conditional symmetry of the linear heat equation, *Dopov. Acad. Sci. Ukraine.*, 1992, N 12, 27–32.
- [10] Buchnev A.A., The Lie group admitted by the equations of the motion of an ideal incompressible fluid, *Dinamika sploshnoy sredy*, Novosibirsk, Institute of Hydrodynamics, 1971, V.7, 212–214.
- [11] Olver P.J., Applications of Lie Groups to Differential Equations, New York, Springer, 1986.
- [12] Popovych H., On reduction of the Euler equations by means of two-dimensional subalgebras, *J. Nonlin. Math. Phys.*, 1996, V.3, N 3–4, 441–446.
- [13] Popovych H., Reduction of the Euler equations to systems in three independent variables, *Dopov. Acad. Sci. Ukraine*, 1996, N 8, 23–29.
- [14] Andreev V.K., Kaptsov O.V., Pukhnachev V.V. and Rodionov A.A., Application of Group Theoretical Methods in Hydrodynamics, Novosibirsk, Nauka, 1994.
- [15] Ludlow D.K., Clarkson P.A. and Bassom A.P., Nonclassical symmetry reductions of the three-dimensional incompressible Navier–Stokes equations, *J. Phys. A: Math. Gen.*, 1998, V.31, 7965–7980.
- [16] Danilov Yu.A., Group properties of the Maxwell and Navier–Stokes equations, Preprint, Acad. Sci. USSR, Kurchatov Institute of Atomic Energy, Moscow, 1967.
- [17] Bytev V.O., Group properties of the Navier–Stokes equations, *Chislennyye metody mekhaniki sploshnoy sredy*, Comp. Center, Siberian Dep. Acad. Sci. USSR, Novosibirsk, 1972, V.3, N 4, 13–17.
- [18] Fushchych W.I. and Popovych R.O., Symmetry reduction and exact solution of the Navier–Stokes equations, *J. Nonlin. Math. Phys.*, 1994, V.1, N 1, 2, 75–113, 158–188.